Caustic Skeleton

&

Cosmic Web

Rien van de Weijgaert & Job Feldbrugge

IHP cosmology workshop “Analytics”, 21 Sept. 2018 collab: Job Feldbrugge, Johan Hidding, Sergei Shandarin, Joost Feldbrugge

Caustic Skeleton

&

Cosmic Web

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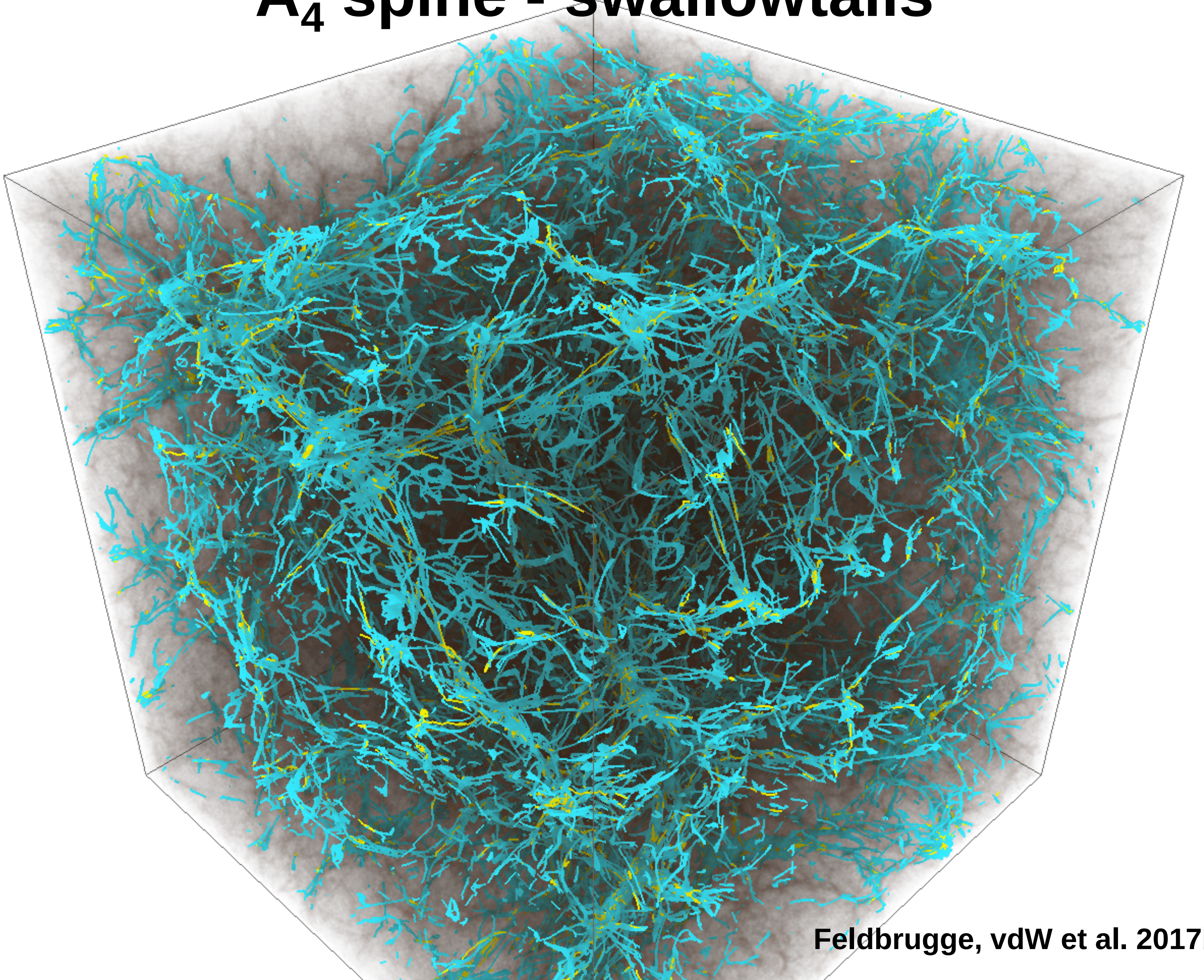
Skeleton (3D) Cosmic Web:

A_4 spine - swallowtails



Feldbrugge, vdW et al. 2017b

- Feldbrugge, vdW, Hidding & Feldbrugge,
  2018, JCAP, 05, 027

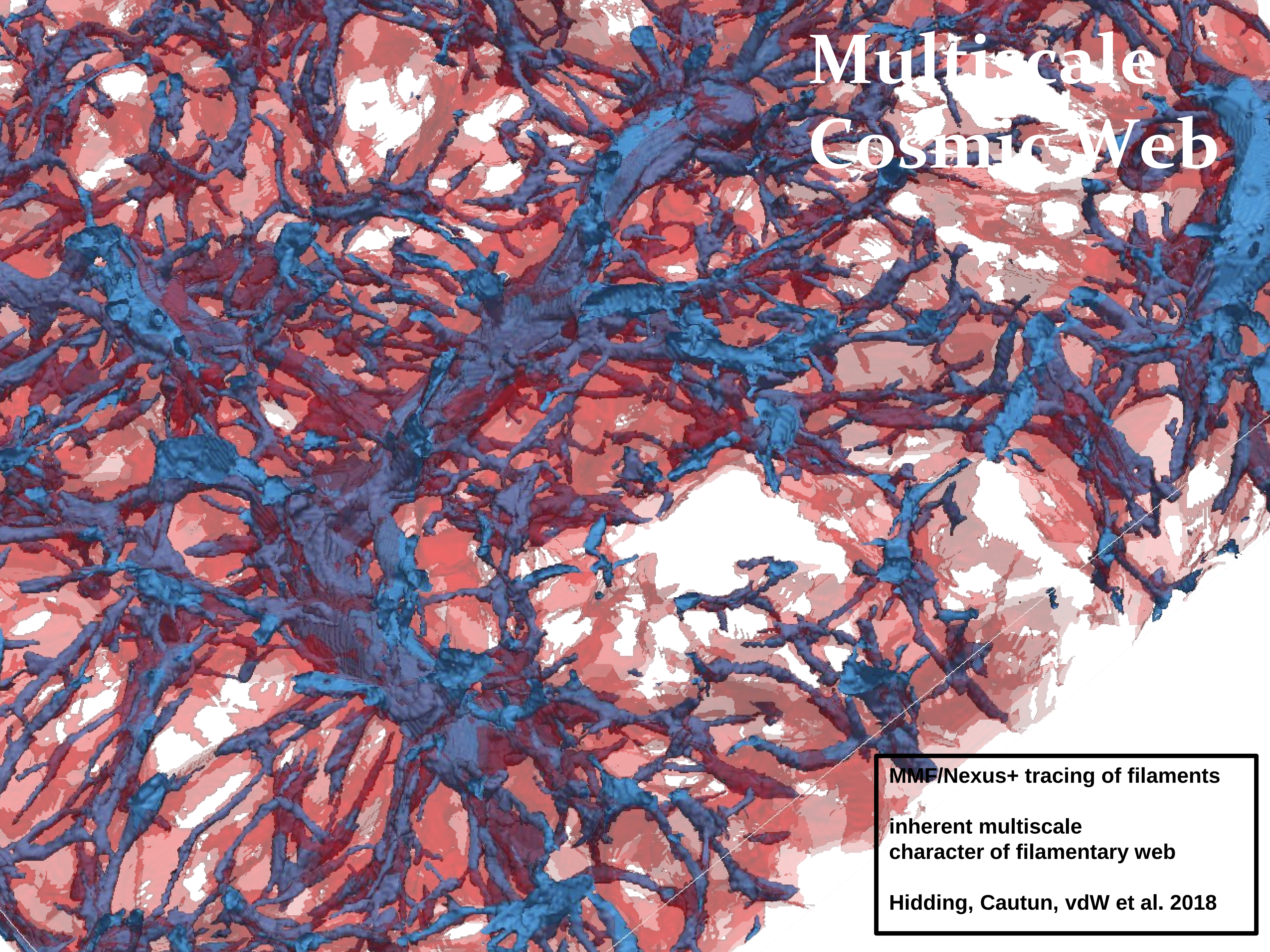
- Feldbrugge, vdW, Hidding & Feldbrugge,
  2018, MNRAS, in prep.

- Hidding, Shandarin, vdW
  2014, MNRAS, 437, 3442

Cosmic Web

Structure & Connectivity

Multiscale Cosmic Web



Cosmic Web Characteristics

- **anisotropic structure:**

  - filaments dominant structural feature - elongated
  - sheets/walls - flattened

- **multiscale nature**

  - structure on wide range of scales
  - structures have wide range of densities

- **overdense-underdense asymmetry**

  - voids: underdense, large & roundish
  - filaments & walls: overdense, flattened/elongated
  - clusters: dense, massive & compact nodes

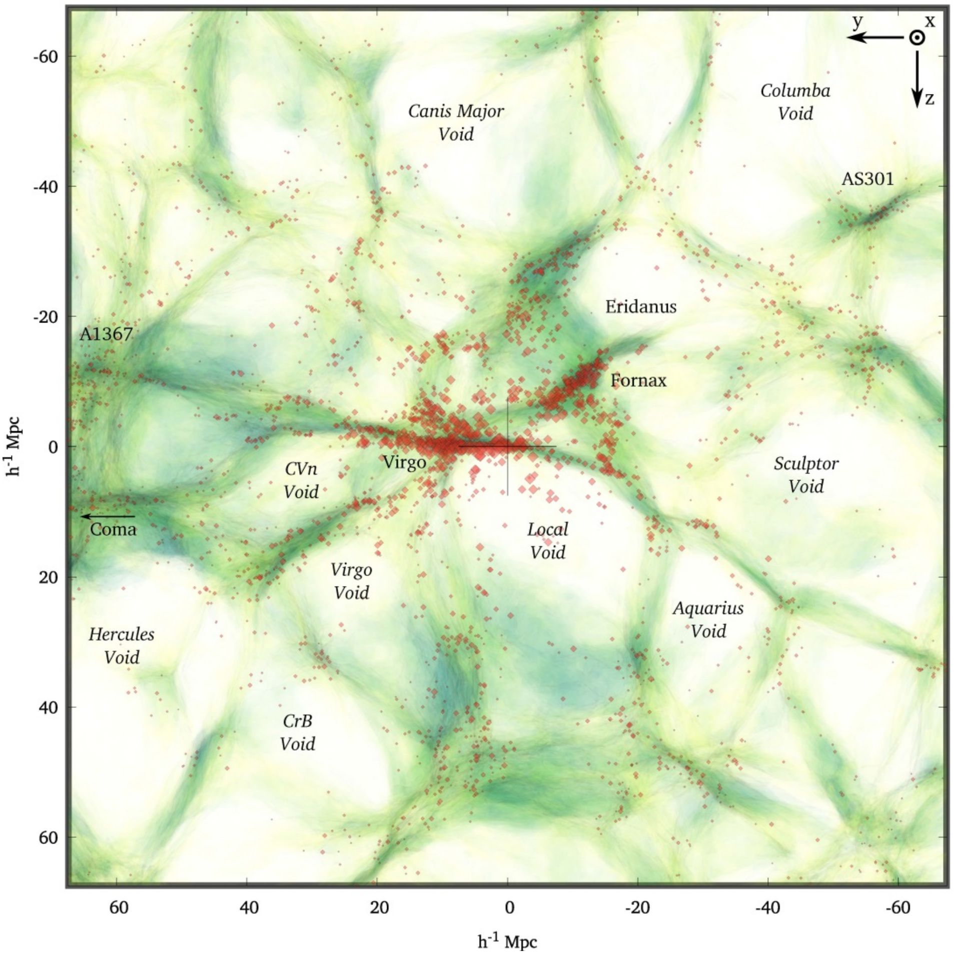
- **complex spatial connectivity**

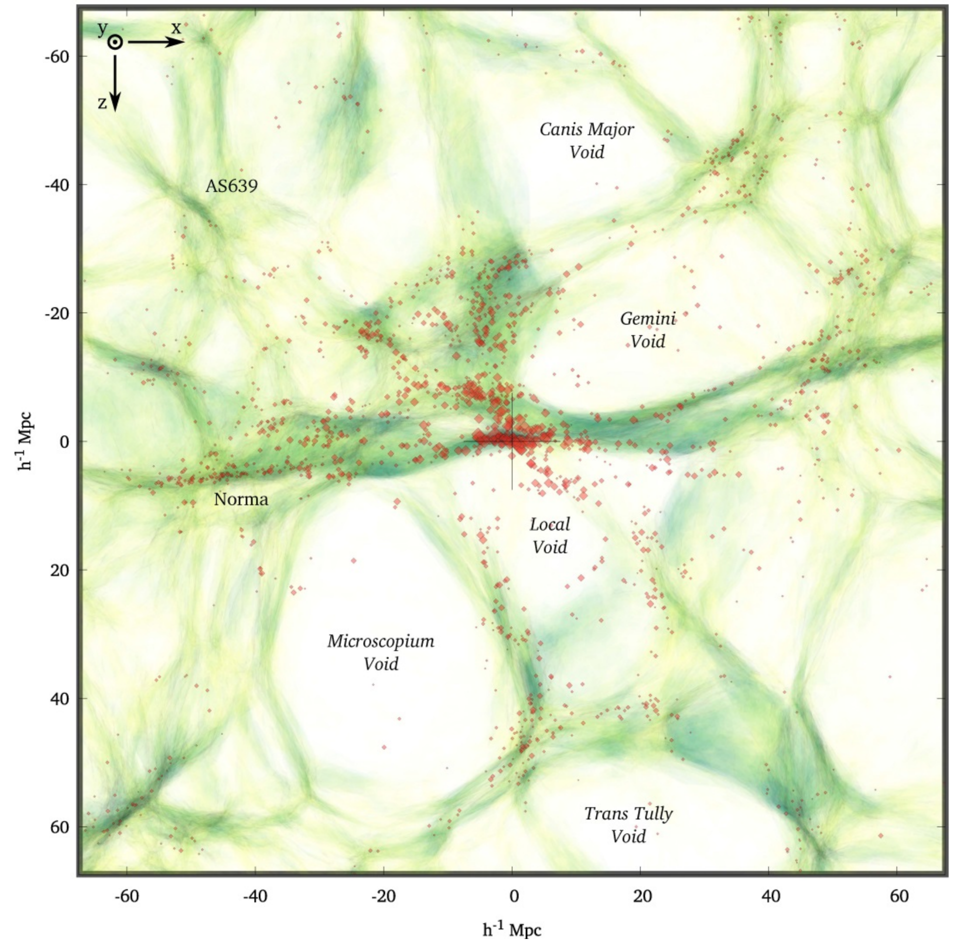
  - all structural features connected in a complex, multiscale weblike network

Void Population Local Universe

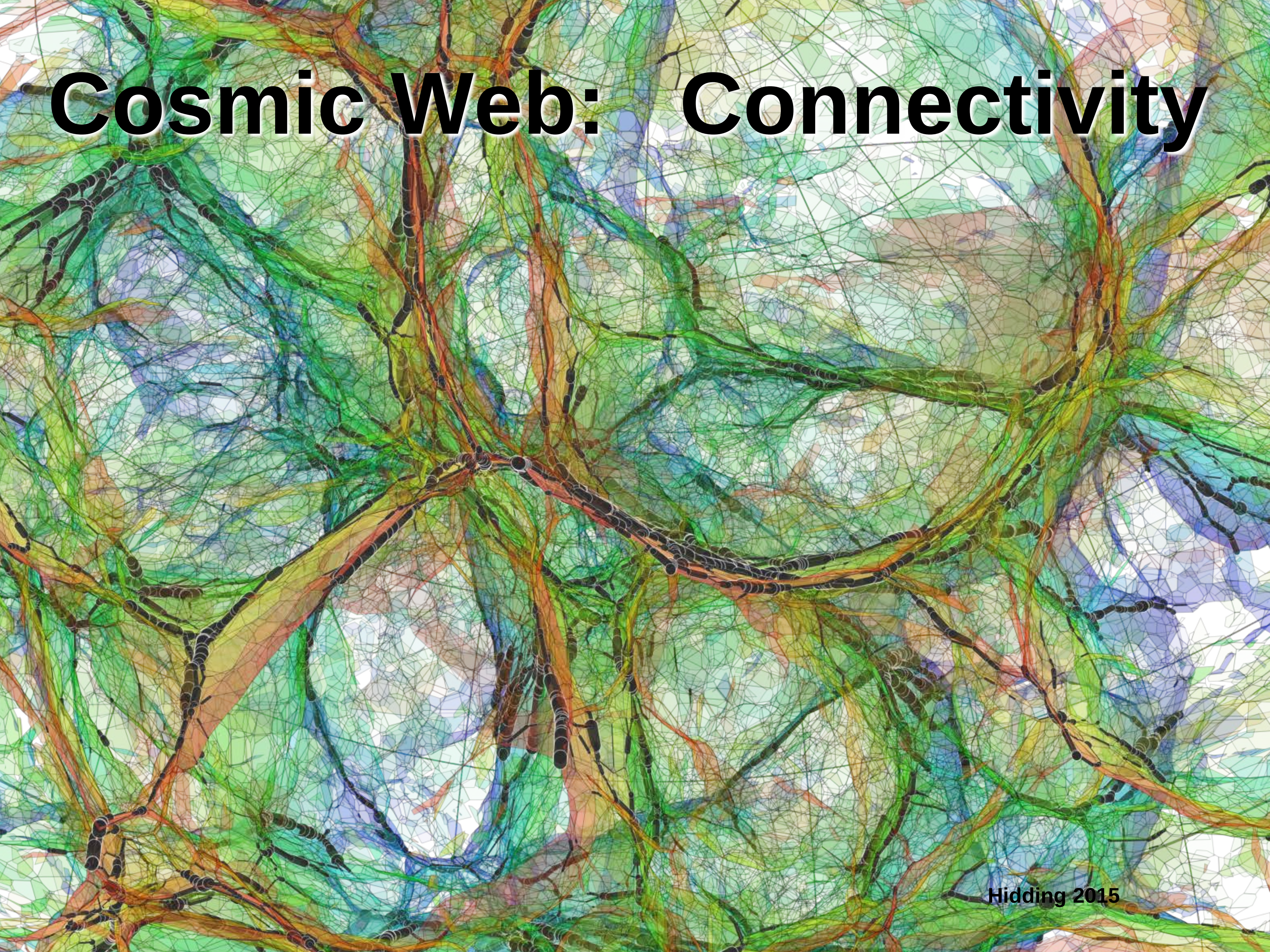
mean KIGEN-adhesion reconstruction (2MRS)

Hidding, Kitaura, vdW & Hess 2016/2017





Cosmic Web: Connectivity

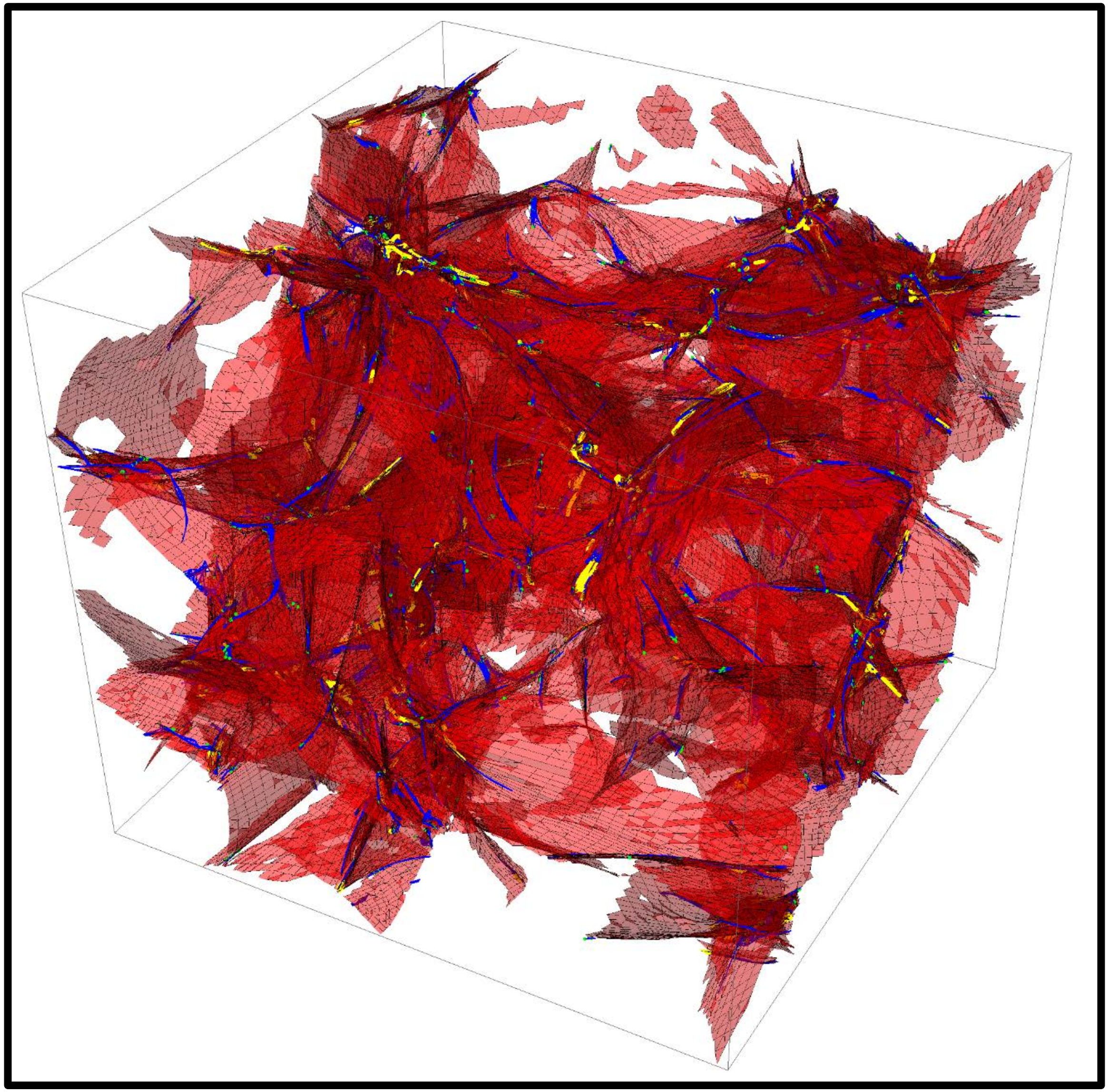


Cosmic Web Skeleton:

Phase Space Dynamics,

Spatial Structure & Connectivity

Cosmic Web - Caustic Skeleton



Cosmic Web: **Caustic features**

⇩

Deformation field (Gaussian initial density field)

Caustic structure

Red sheets - cusps A3 singularities - walls

Blue lines - swallowtail A4 sing. - filaments

Green dots - butterfly A5 sing. - nodes

Yellow lines - D4 umbilics - filaments

Caustic connectivity:

Topology of the deformation field

Cosmic Web

the Phase-Space View

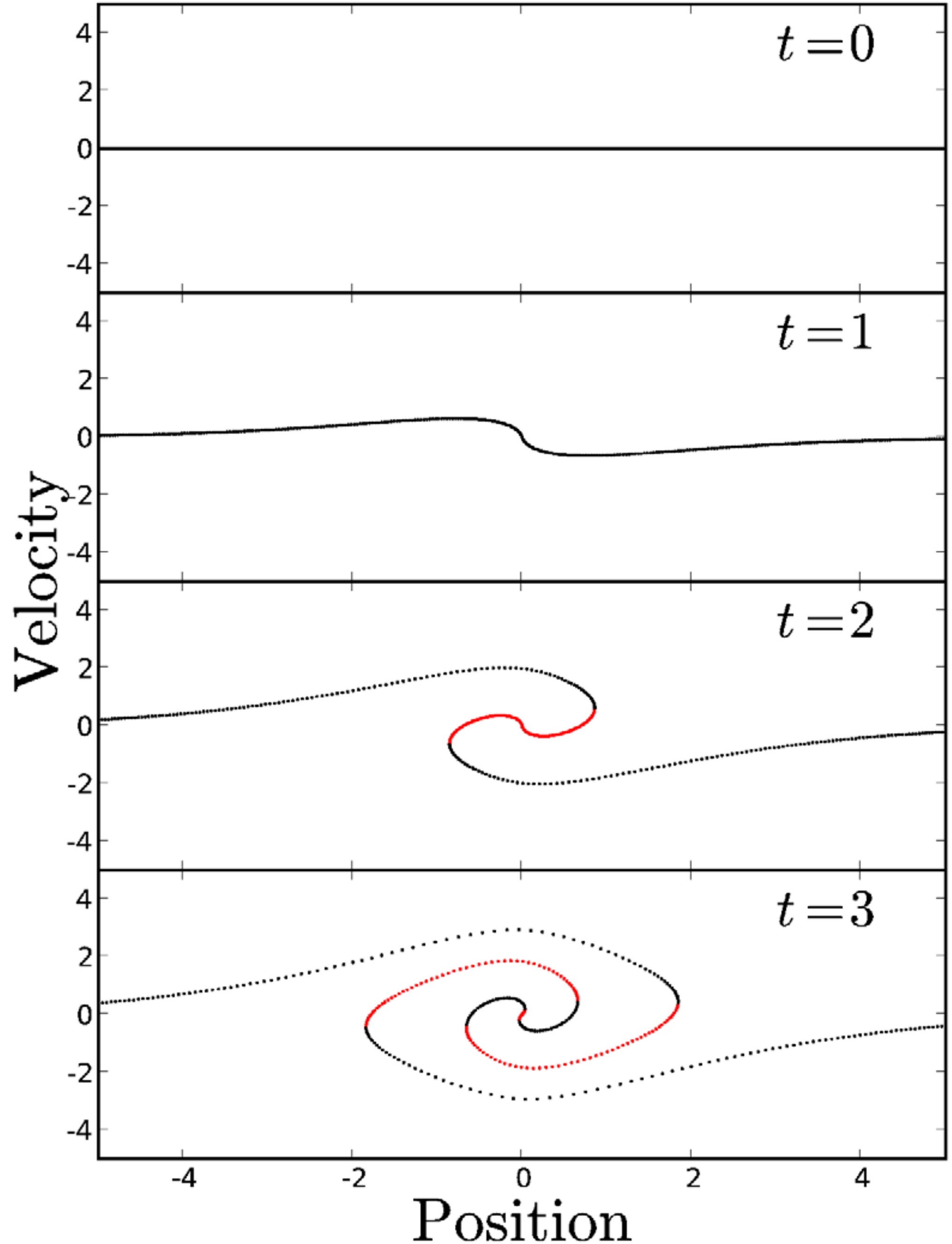
Phase Space Evolution

Dark Matter Phase Space sheet:

3-D structure projection of a folding DM phase space sheet In 6-D phase space

- **Shandarin 2010, 2011**
- **Neyrinck et al. 2011, 2012 Origami**
- **Abel et al. 2011**

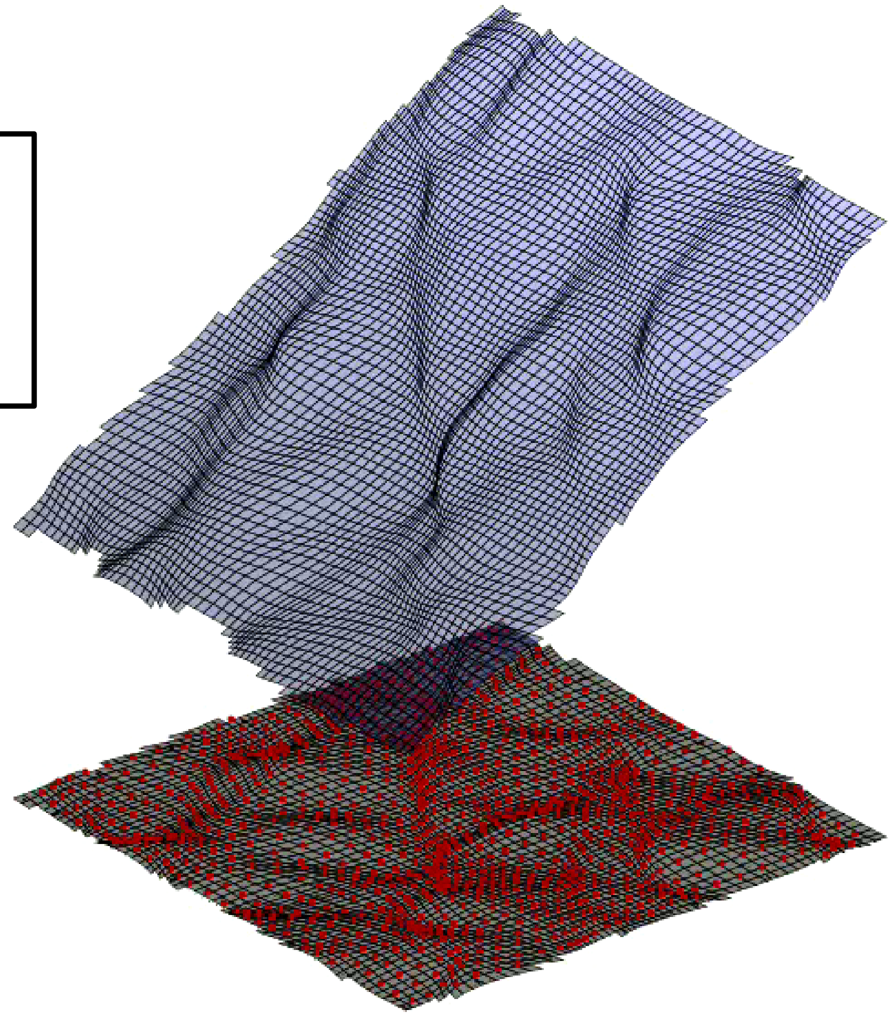
Evolving matter distribution in position-velocity space – 1D



Phase-Space Evolution

Dynamical Evolution:

folding the
phase-space sheet {q,x}



Lagrangian coordinate

Eulerian plane

Hidding 2014

Phase-Space Evolution

Dynamical Evolution:

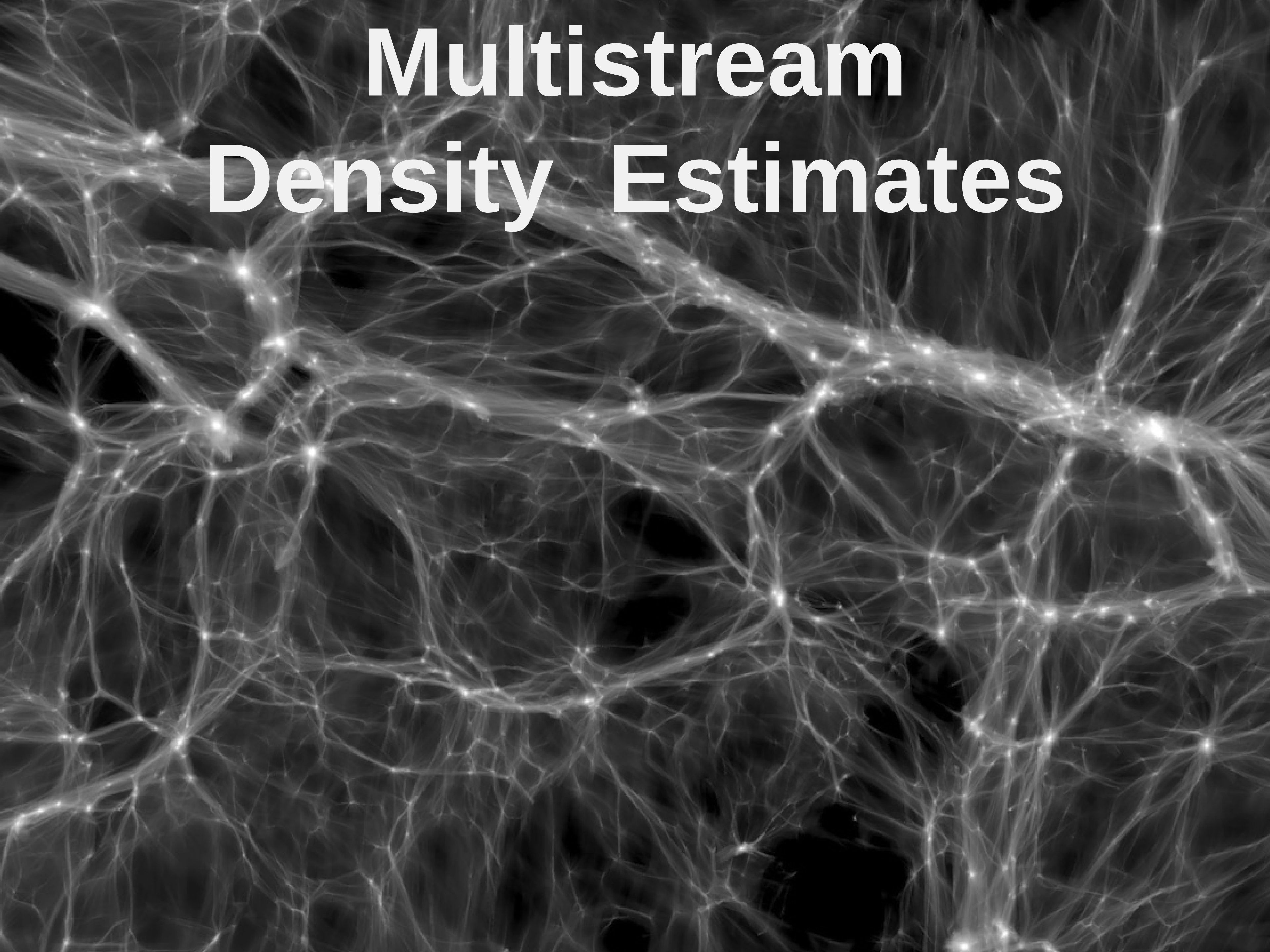
WDM simulation
projected density field

- **Cosmic Web formation**
- **Multistream regions**
- **Caustic structure**

Simulation:
R. Angulo & O. Hahn

Courtesy:
R. Angulo & O. Hahn

Multistream Density Estimates



Cosmic Web Stream Density

Translation towards Multi-D space:

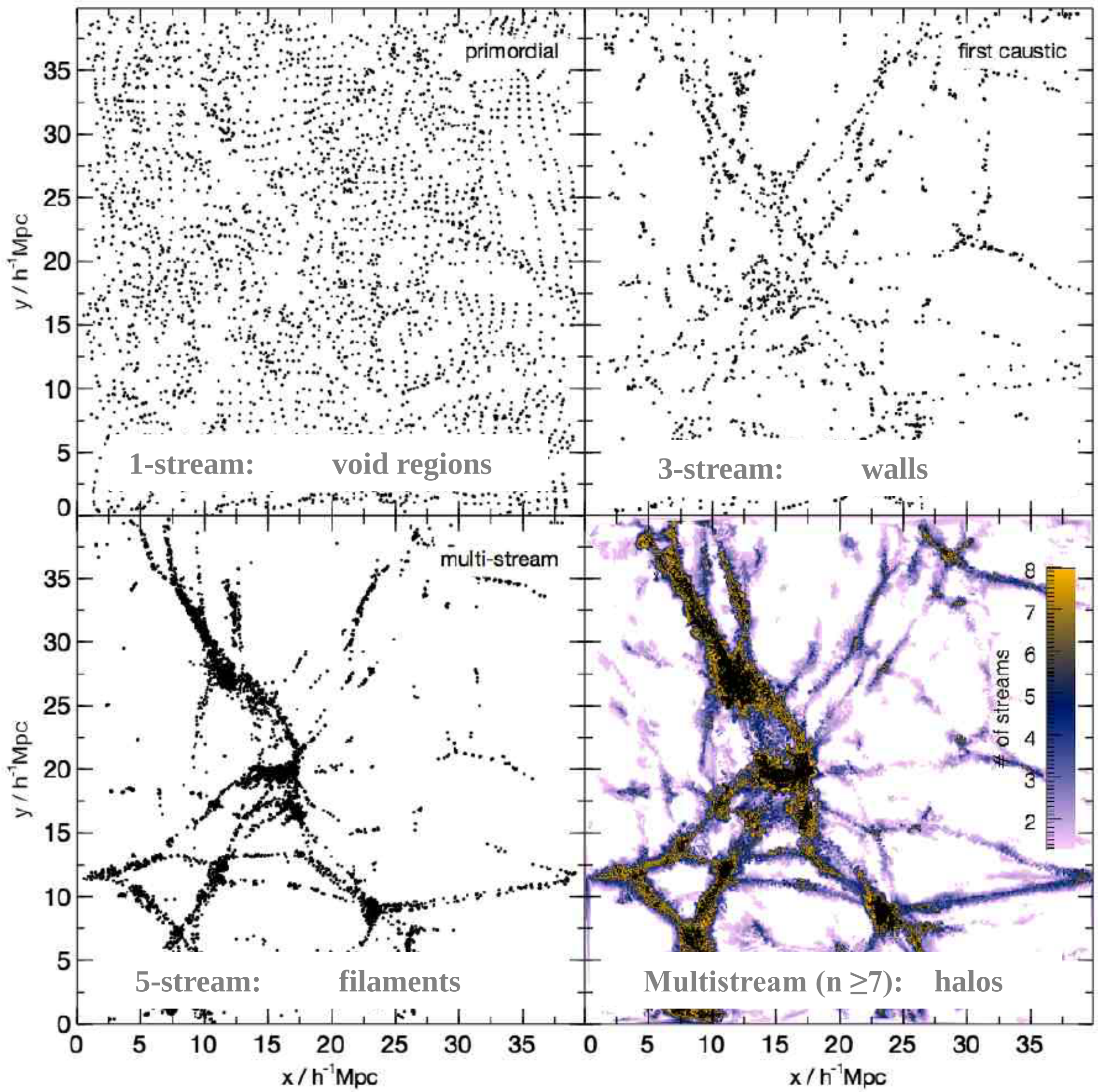
Density of dark matter streams:

- **# phase space folds**

=

 # locally overlapping tessellation cells

Shandarin 2012
Abel, Hahn & Kaehler 2012
Falck, Neyrinck et al. 2012



Cosmic Web Dynamics:

tidal weaving the Cosmic Tapestry

Formative agent of the Cosmic Web:

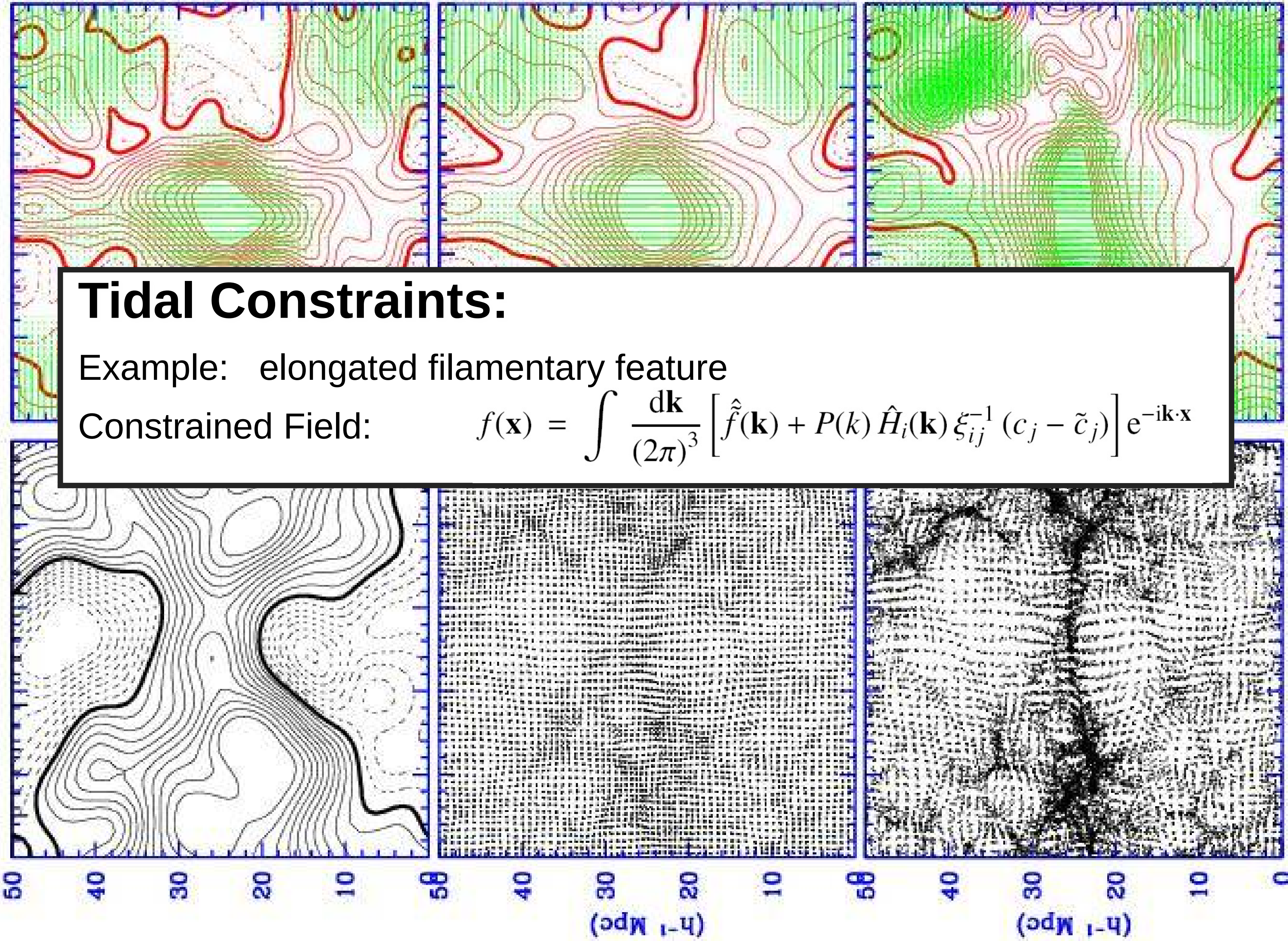
Tidal strain induced my the Megaparsec Matter Distribution:

- anisotropic collapse of structures
- connection clusters-filaments:

clusters main agent for stretching filaments

$$T_{ij}(\vec{r},t) = \frac{3\Omega H^2}{8\pi} \int d\vec{x}\ \delta(\vec{x},t) \left\{ \frac{3(x_i - r_i)(x_j - r_j) - |\vec{x} - \vec{r}|^2 \delta_{ij}}{|\vec{x} - \vec{r}|^5} \right\}$$

$$- \ \frac{1}{2}\Omega H^2 \delta(\vec{r},t)\ \delta_{ij}$$

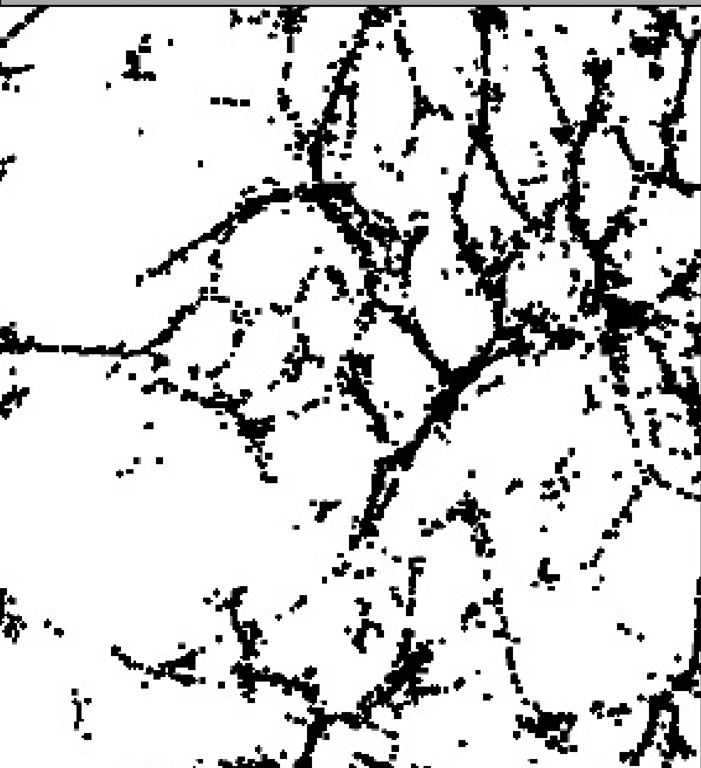


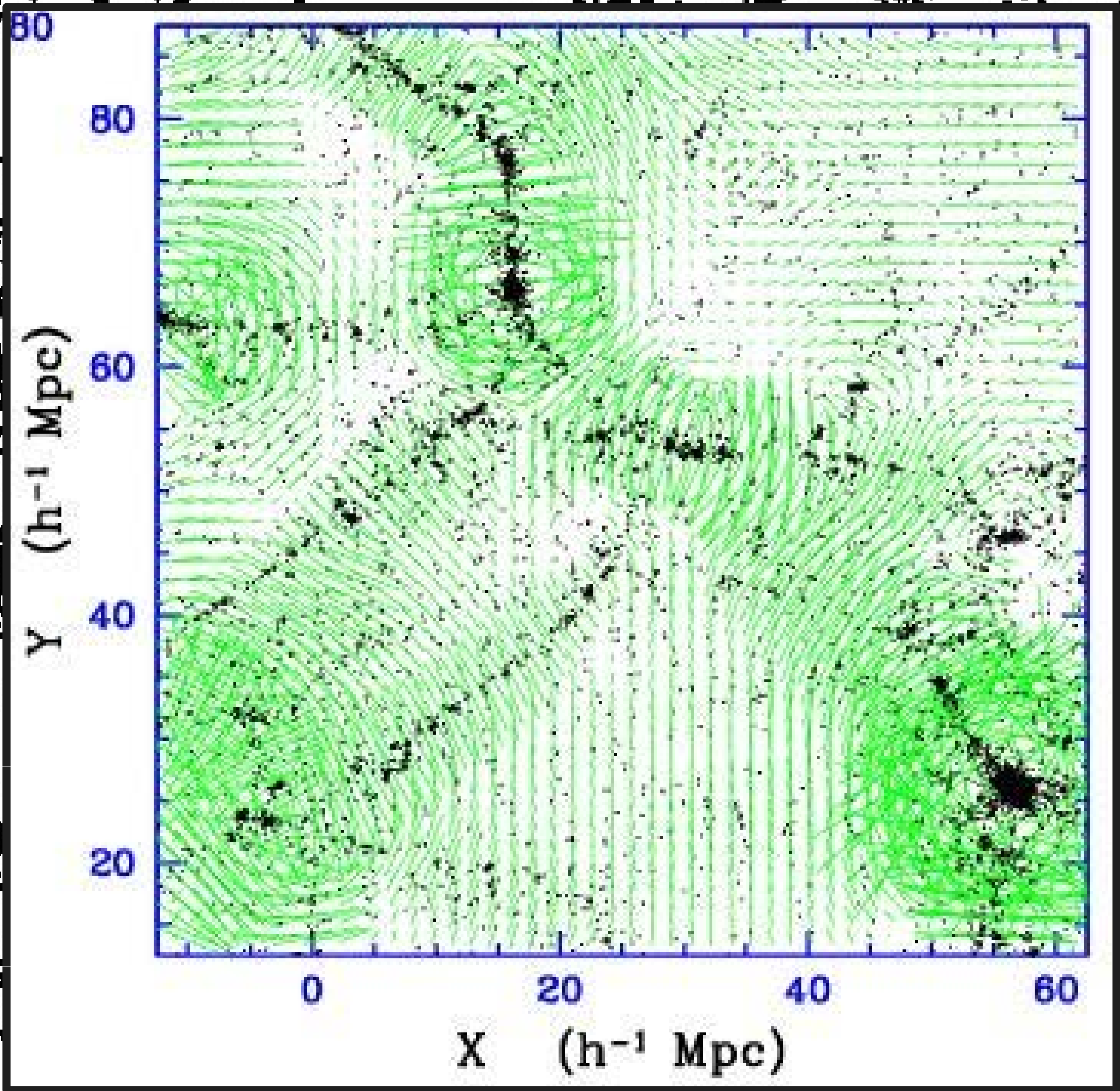
Tidal Constraints:

Example: elongated filamentary feature

Constrained Field: $$f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \left[\hat{\tilde{f}}(\mathbf{k}) + P(k)\, \hat{H}_i(\mathbf{k})\, \xi_{ij}^{-1}\, (c_j - \tilde{c}_j) \right] e^{-i\mathbf{k}\cdot\mathbf{x}}$$

Tidal Shaping of the Cosmic Web

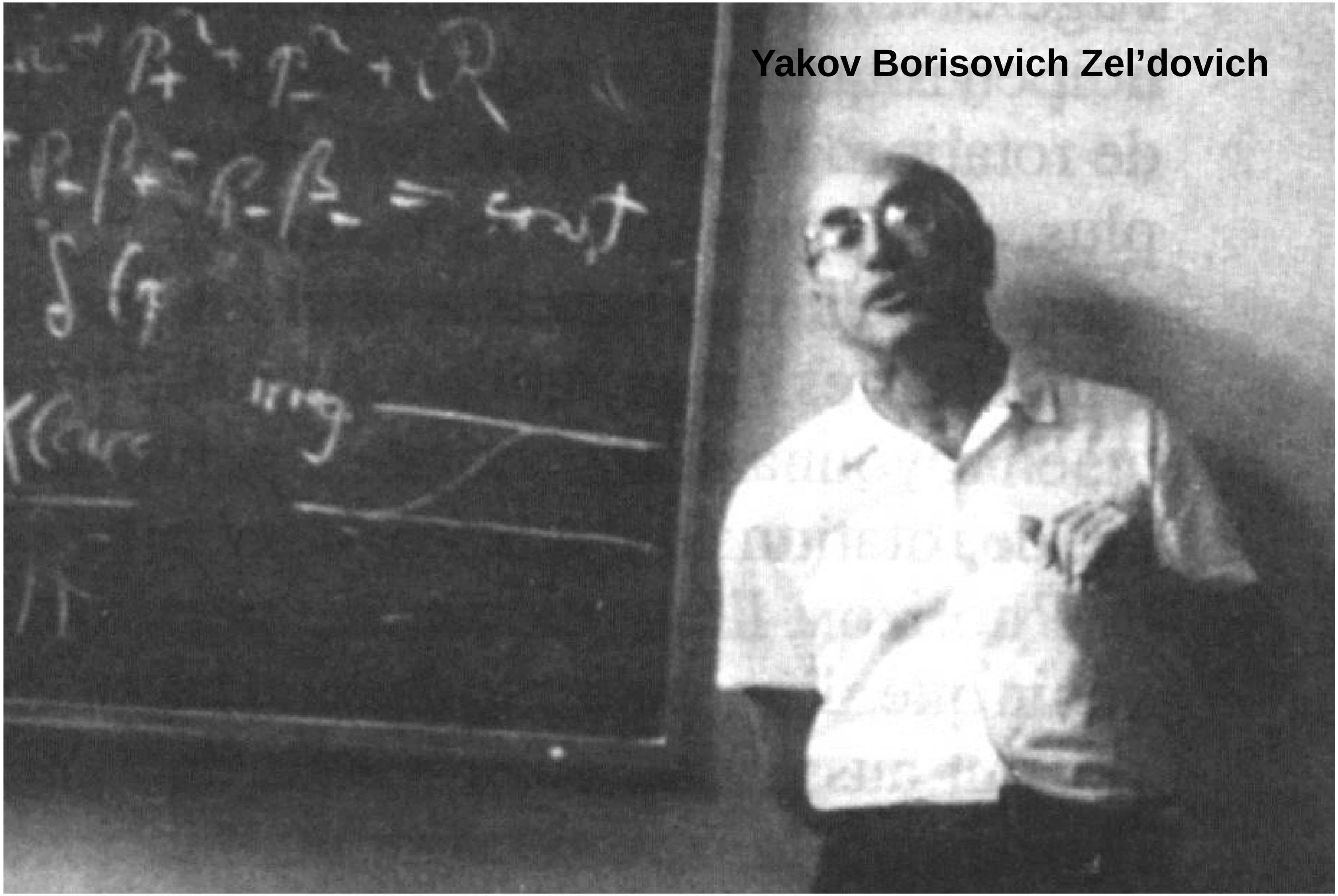




Tidal Forces shape the Cosmic Web

Phase-Space Evolution:

Zeldovich & Deformation


Yakov Borisovich Zel’dovich

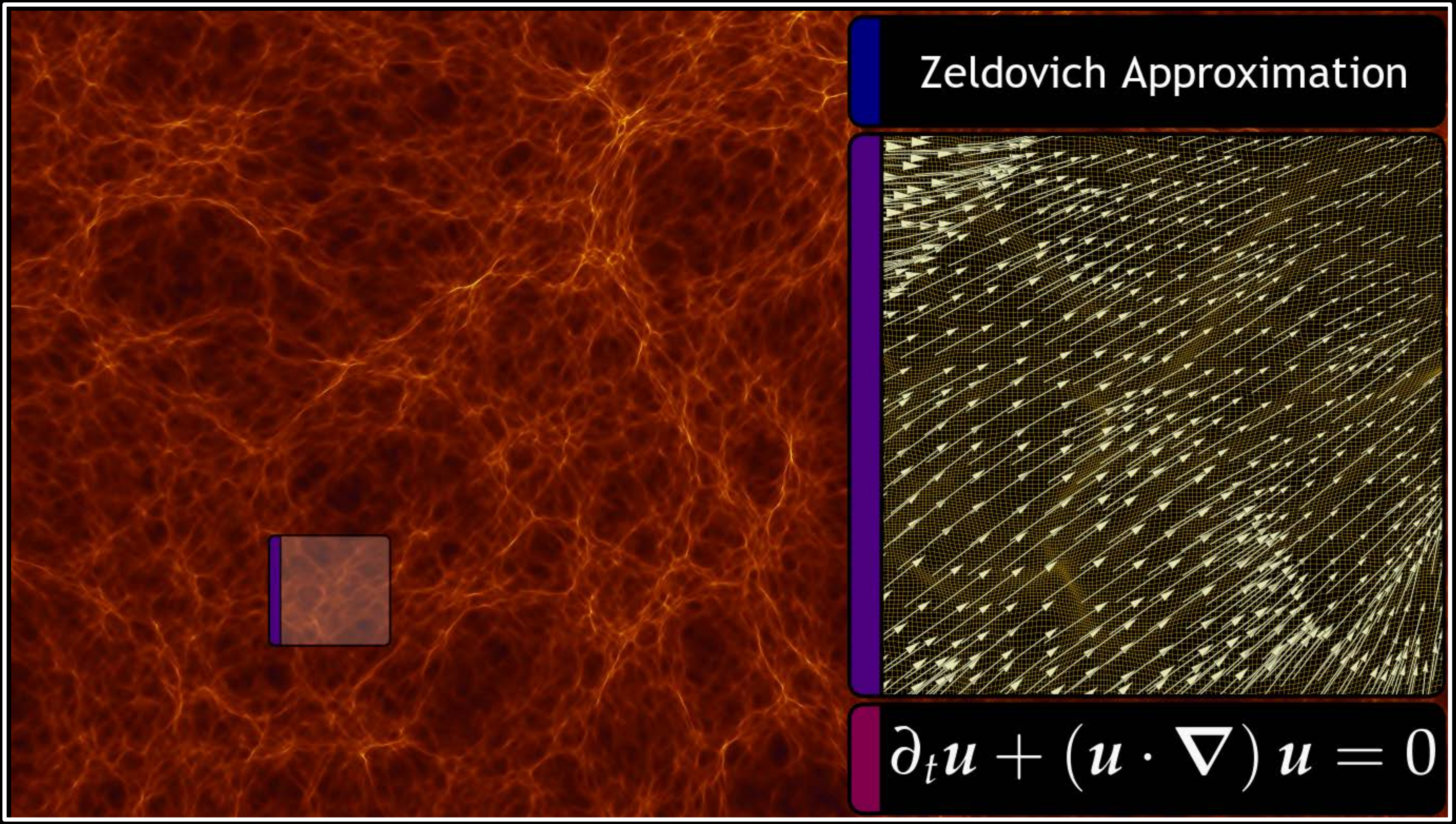
Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi_{lin}\left(\vec{q}\right)$$

Zel’dovich Approximation



Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$

structure of the cosmic web determined by the spatial field of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

Phase-Space Evolution:

Catastrophes & Caustics

Zel'dovich Formlism:
Streaming & Caustics

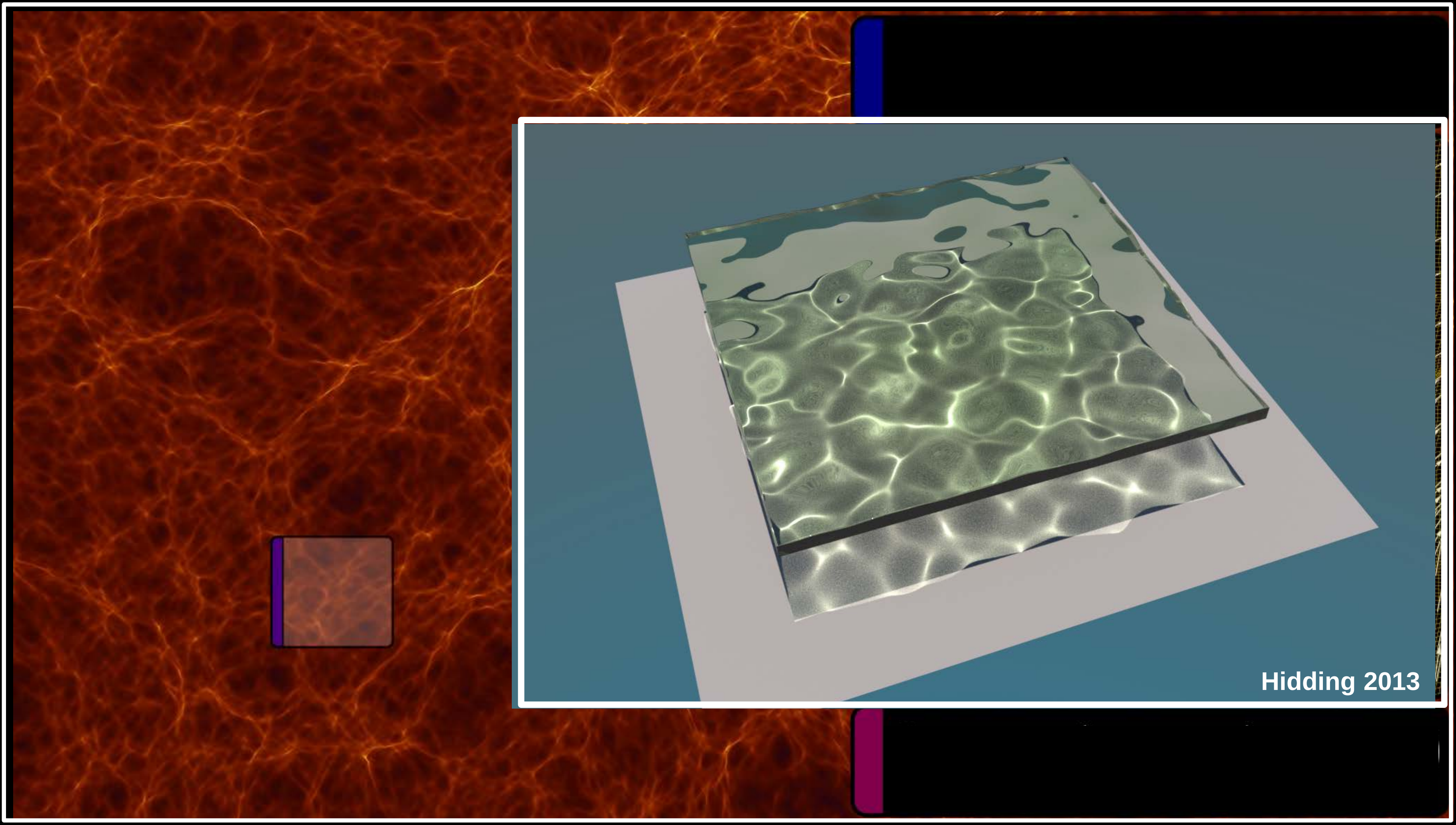


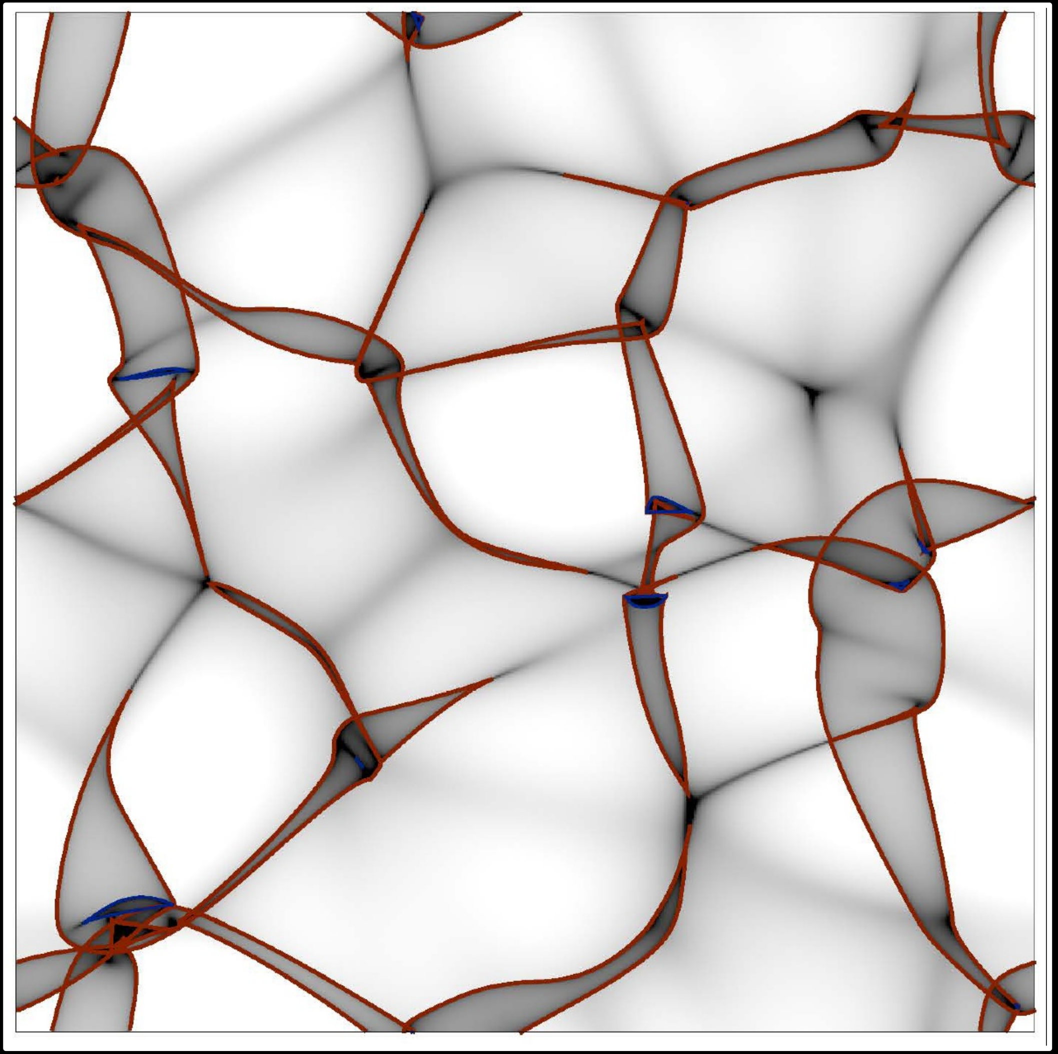
Illustration of the formation of caustics due to streaming paths of light through deforming medium

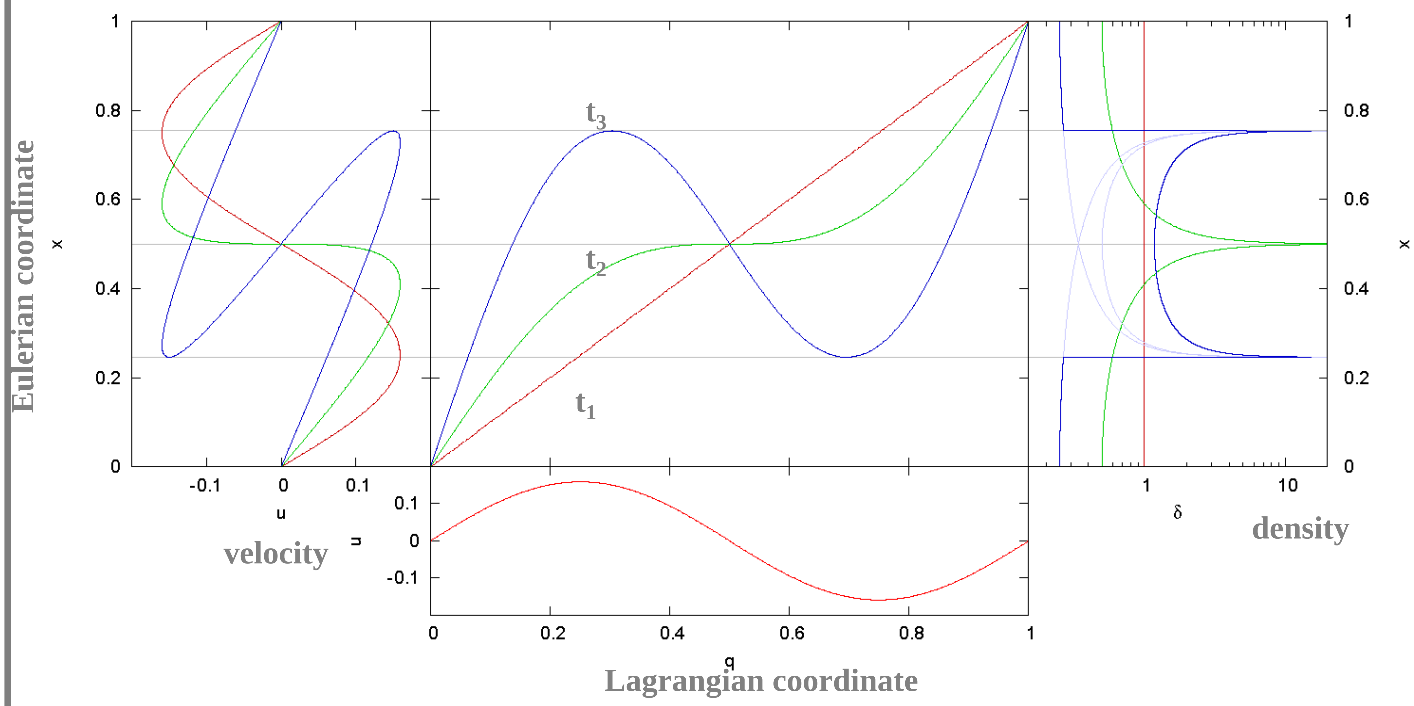
Zel'dovich Approximation

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi_{lin}(\vec{q})$$





Zeldovich Formalism:

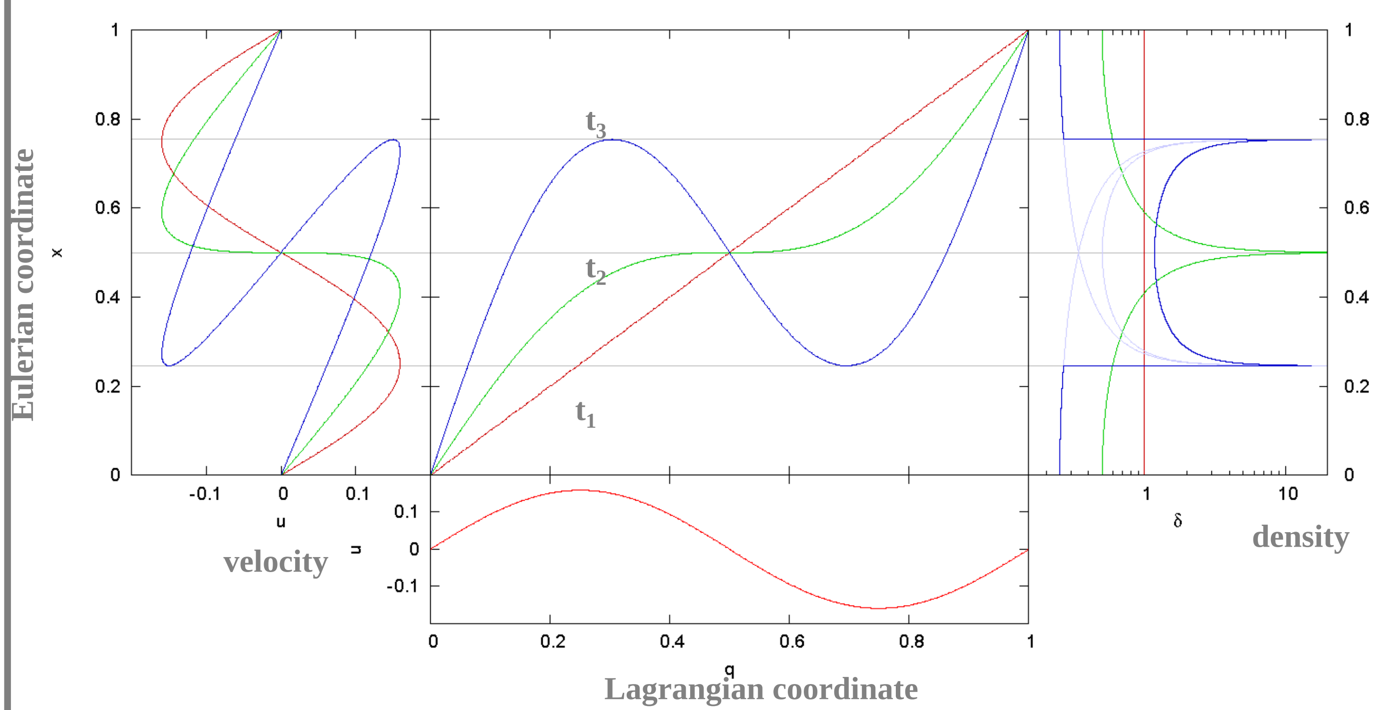
Singularities

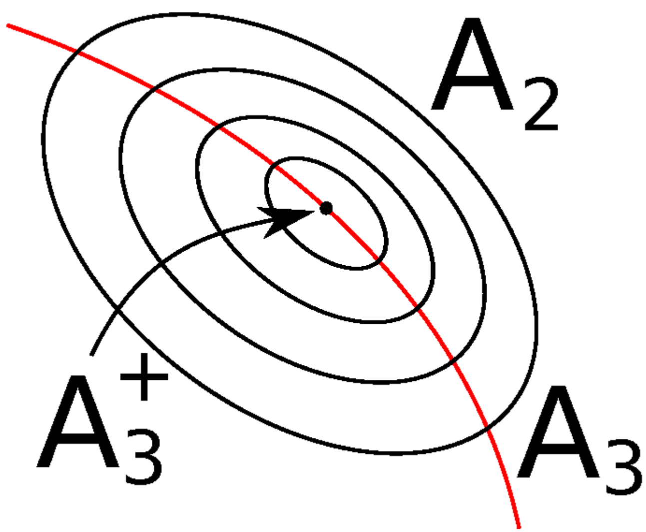
$$\vec{x}(\vec{q},t) = \vec{q} - D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} : \ \lambda_1, \lambda_2, \lambda_3$$

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$

Caustic Formation:

Folds & Cusps



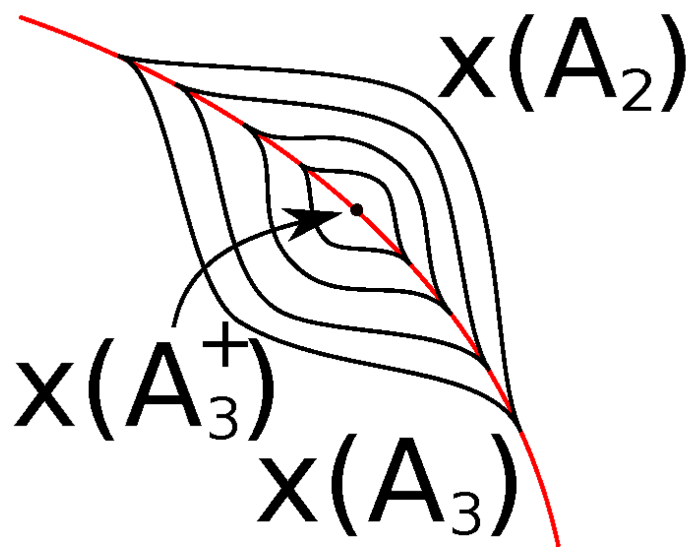


Lagrangian space: A_2 contours

$$\rho(\vec{q},t) = \infty$$

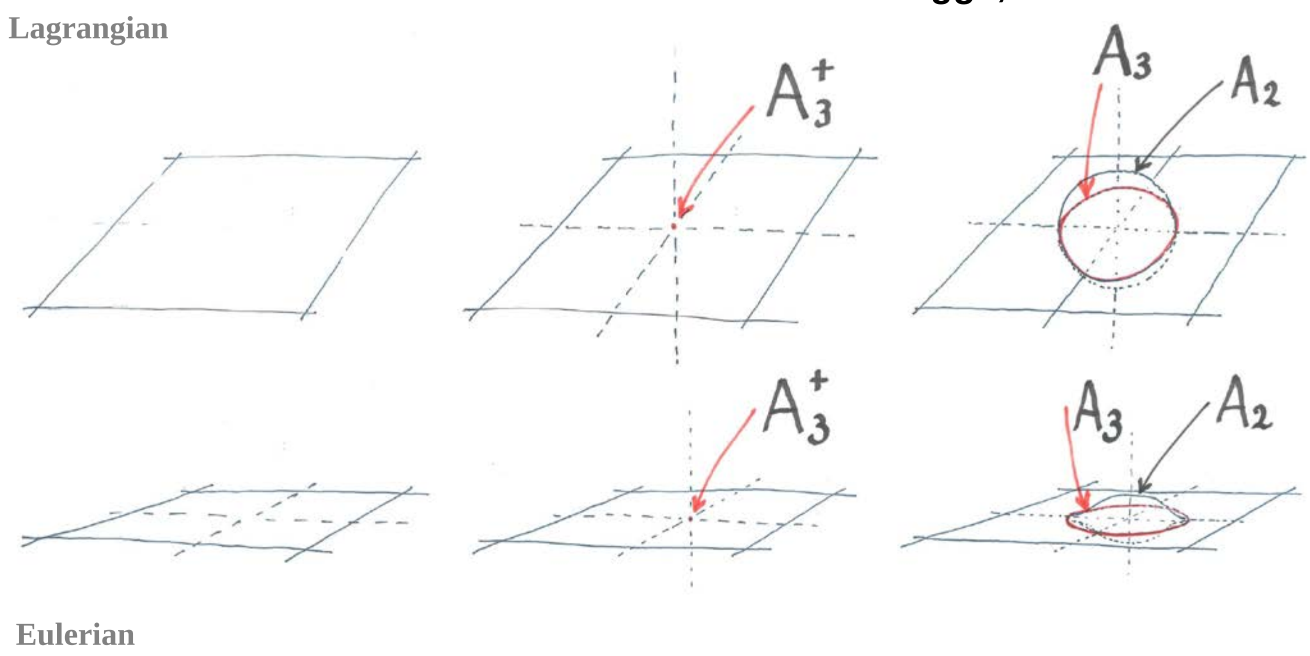
$$\Uparrow$$

$$D(t)\lambda_i = 1$$



Eulerian space: folds & cusps

Feldbrugge, vdW et al. 2017



Emergence of A_3 cusps around A_2 folds

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

(also see Zeeman, Thom)

singularities

D=1

A: λ_1

D=2

A: λ_1

D: λ_1, λ_2

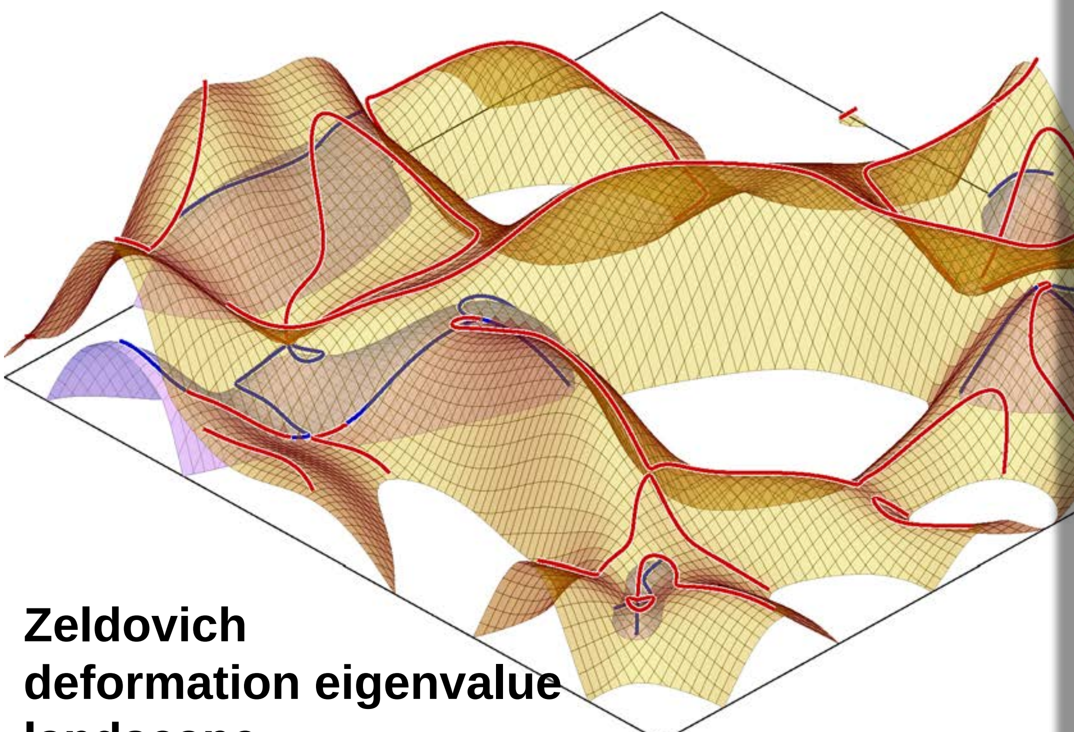
D=3

A: λ_1

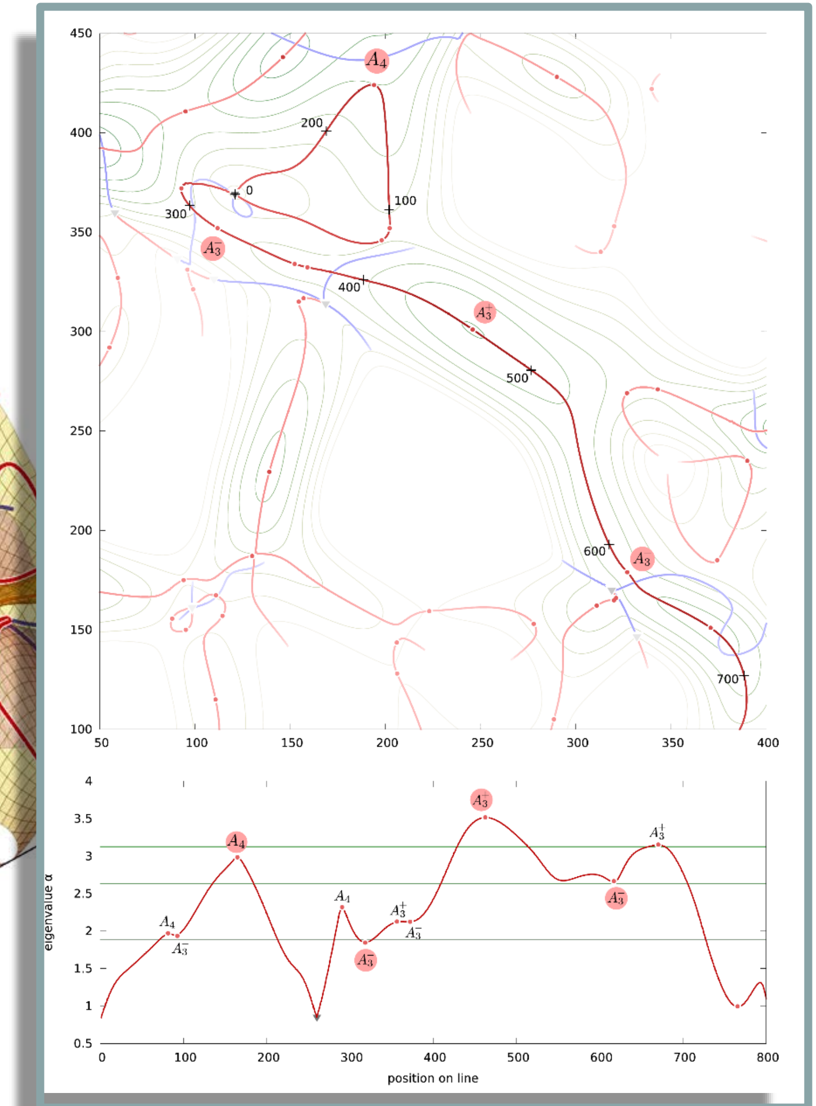
D: λ_1, λ_2

E: λ_1, λ_2, λ_3

Singularities & Catastrophes:
Deformation Field

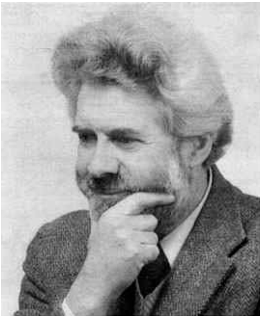


Zeldovich deformation eigenvalue landscape



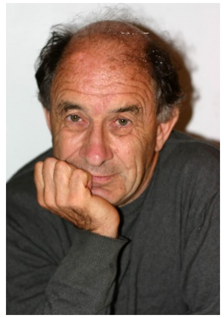
Hidding, Shandarin & vdW 2014

Leaders of Catastrophe



E. Zeeman

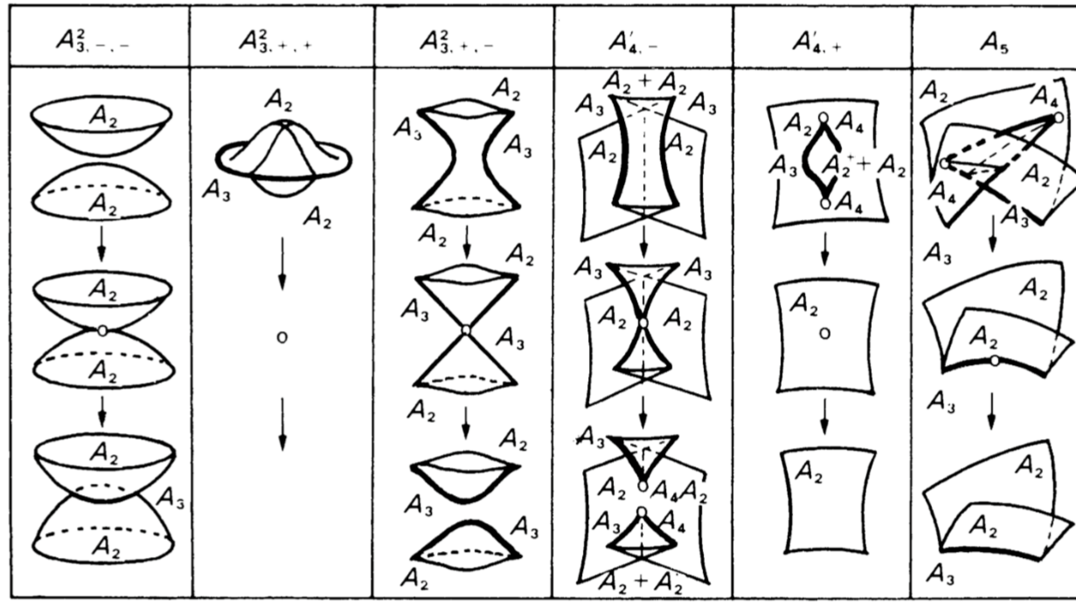
R. Thom

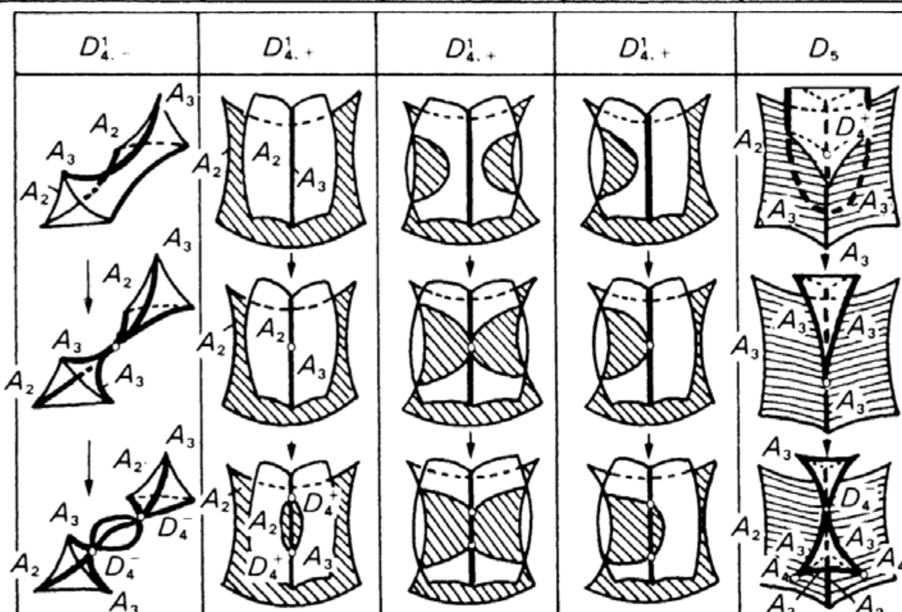


V.I. Arnol'd

Arnold V.I.   (and others):
Caustic classification on basis of Normal Forms:

$$F(q_1) = q_1^{\,5} + \lambda q_1^{\,3} + \mu q_1^{\,2}$$





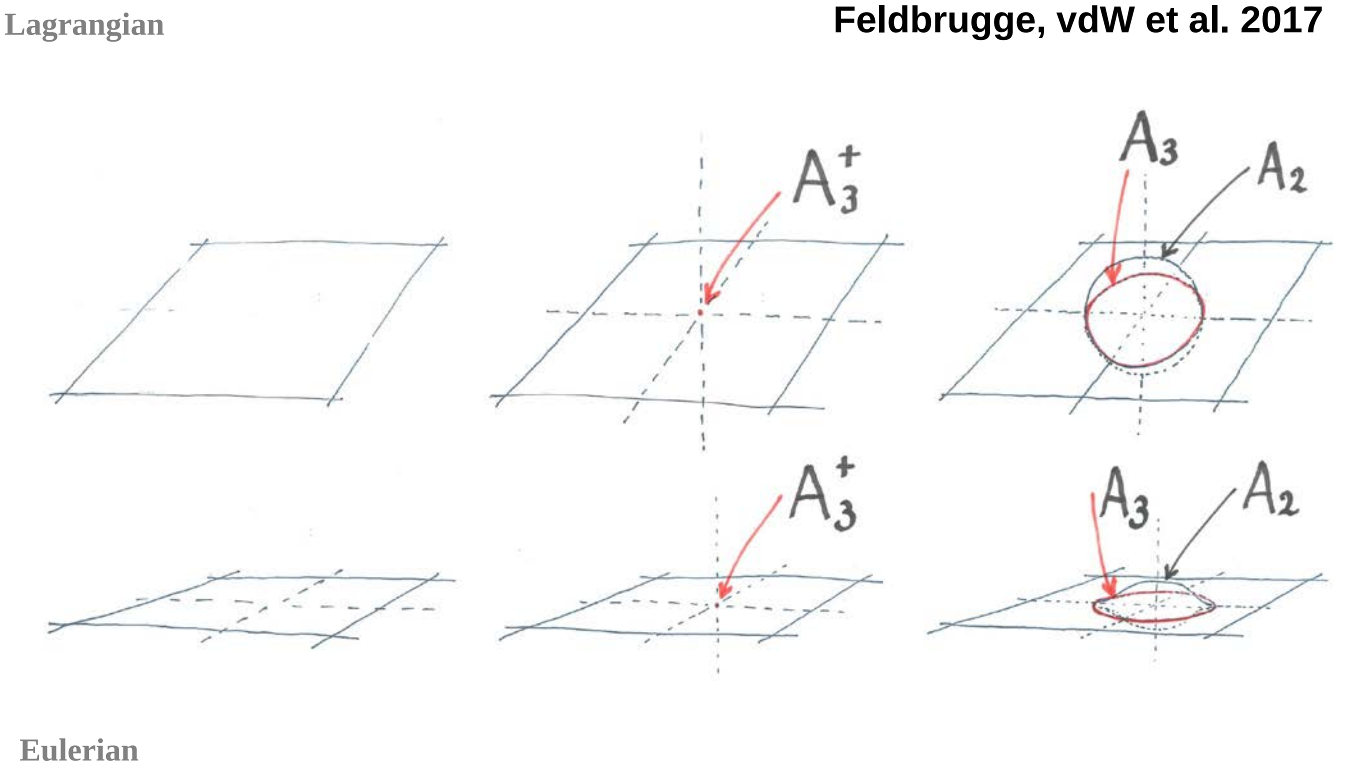
Arnold V.I., 1986,
Catastrophe Theory, Springer

Arnold V.I., Shandarin S.F., Zeldovich Ya.B., 1982
The Large Scale Structure of the Universe I. General Properties. One and Two-dimensional models
Geophys. Astrophys. Fluid Dynamics, 20, 1-2

Arnold V.I., 1986,
Evolution of singularities of potential flows in collisionless-free media and the metamorphosis of caustics in three-dimensional space

Caustic Classes

Feldbrugge, vdW et al. 2017



Emergence of A_3 cusps around A_2 folds (3D)

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

(also see Zeeman, Thom)

D=1

A_2: folds
A_3: cusps

D=2

A_2: folds
A_3: cusps
A_4: swallowtail

D_4: umbilics

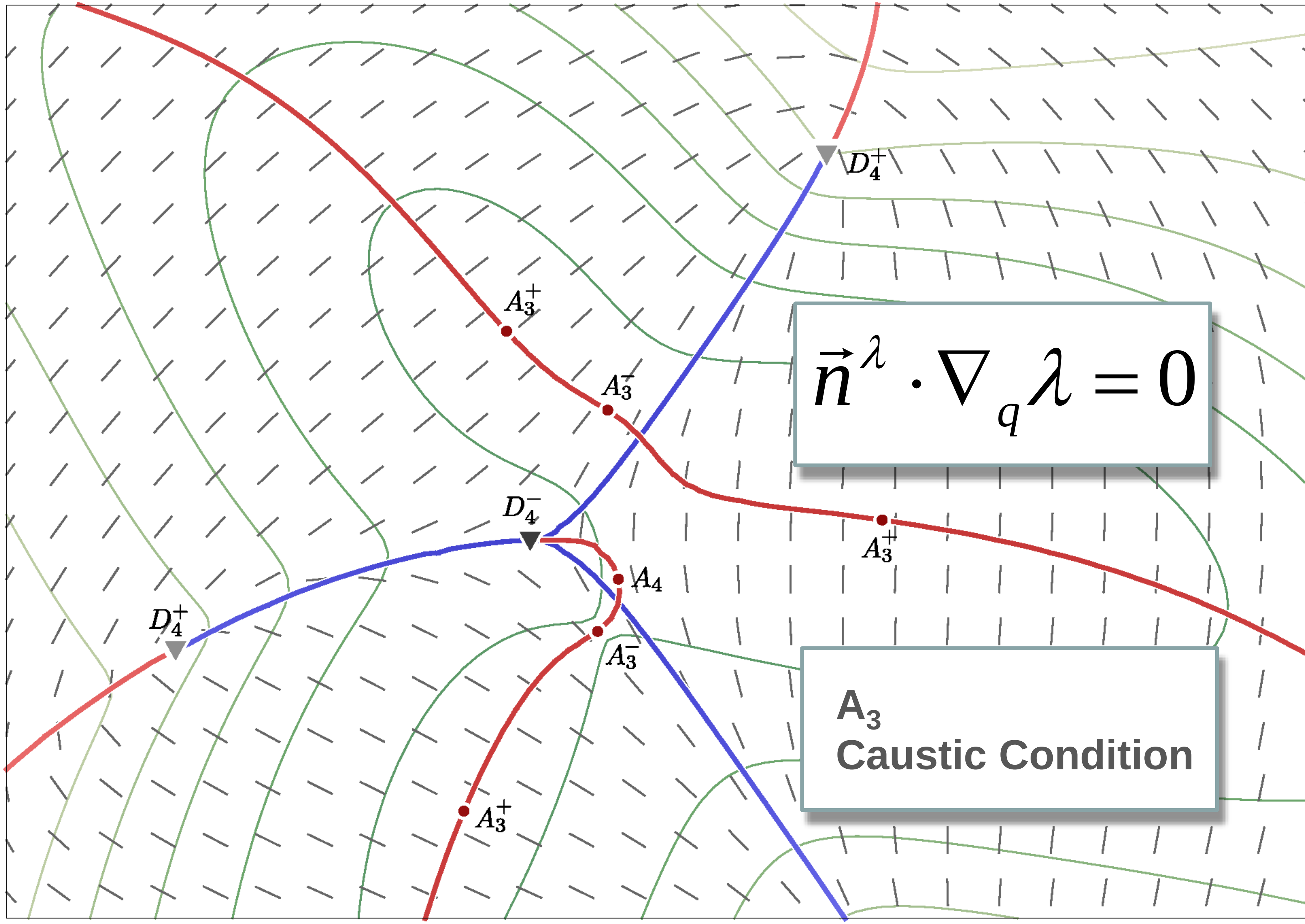
D=3

A_2: folds
A_3: cusps
A_4: swallowtail
A_5: butterfly

D_4: umbilics
D_5

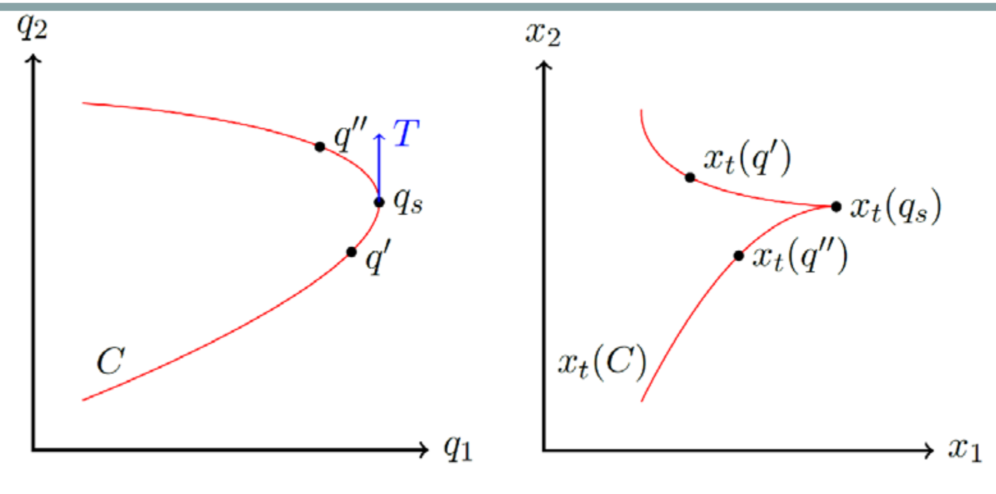
E_5

Caustic Conditions



D_4^+
A_3^+
A_3^-
$\vec{n}^{\lambda} \cdot \nabla_q \lambda = 0$
D_4^-
A_3^+
A_4
D_4^+
A_3^-
A3 Caustic Condition
A_3^+

Caustic Conditions



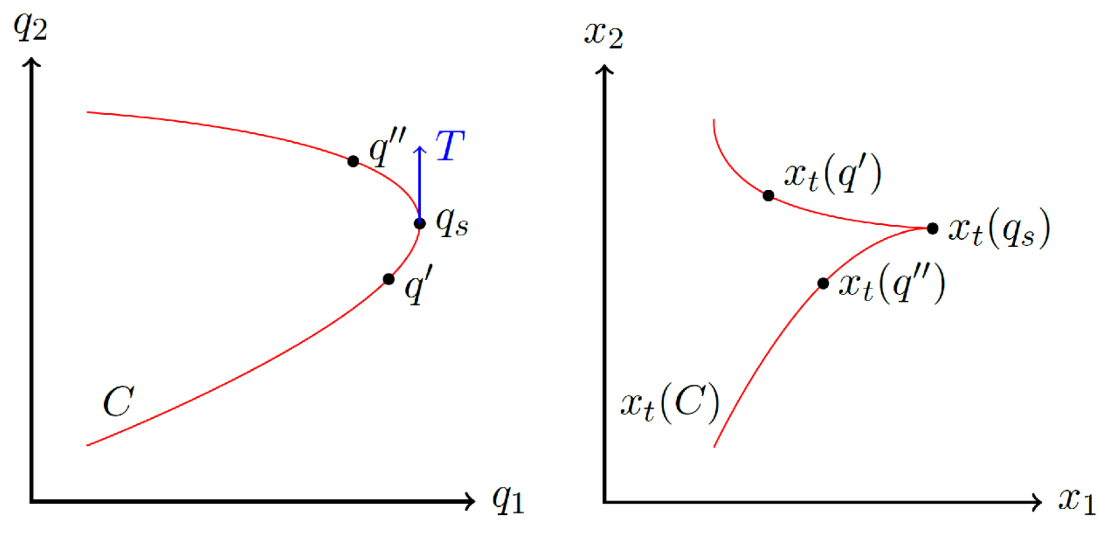
(a) Lagrangian space L (b) Eulerian space E

$$\frac{\Delta x}{\Delta q} = \frac{\|x_t(q') - x_t(q'')\|}{\|q' - q''\|} \to 0 \qquad q', q'' \to q_s$$

$$\begin{cases} \left\| \dfrac{\partial x_t(q)}{\partial q} \cdot T \right\| = 0 \\ \\ T + \dfrac{\partial s_t(q)}{\partial q} \cdot T = 0 \end{cases}$$

$$M = \frac{\partial s_t(q)}{\partial q} \quad \Rightarrow \quad M v_i = \mu_i v_i \quad \Rightarrow \quad M_d = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

Caustic Conditions



(a) Lagrangian space L (b) Eulerian space E

$$T + \frac{\partial s_t(q)}{\partial q} \cdot T = 0$$

$$0 = (I + M) V V^{-1} T = V (I + M_d) V^{-1} T$$

$$V^{-1} = \begin{pmatrix} v^*_{11} & v^*_{12} & v^*_{13} \\ v^*_{21} & v^*_{22} & v^*_{23} \\ v^*_{31} & v^*_{32} & v^*_{33} \end{pmatrix}, \quad v_i \cdot v^*_j = \delta_{ij}$$

$$(I + M_d) V^* T = 0$$

Caustic Conditions

In Lagrangian space L (coordinates q):

A singularity forms in a manifold $\mathrm{M} \subset L$ at location q_s when at q_s,

- the deformation tensor eigenvalue $\mu_i(q_s)$
- the corresponding eigenvector $\vec{v}_i\ (q_s)$

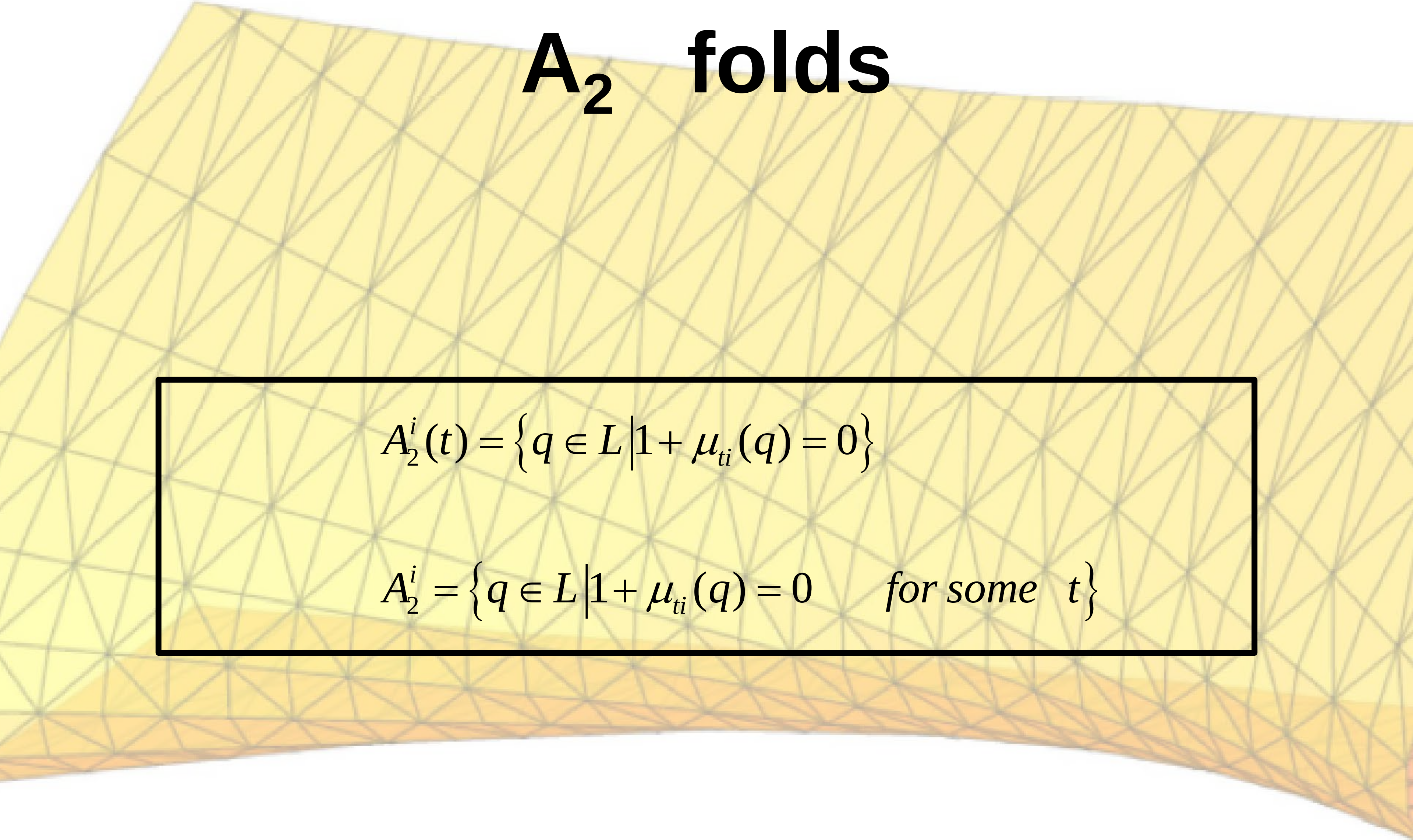
when at least one nonzero tangent vector $\vec{T}$ such that

$$\{1+\mu_i(q_s)\}\ \vec{v}_i^{\,*}(q_s)\cdot\vec{T} = 0$$

NOTE: *Nature of singularity not only dependent on EIGENVALUES* $\mu_i(q_s)$*, but also EIGENVECTORS* $\vec{v}_i\ (q_s)$

Feldbrugge, vdW et al. 2017a

Caustic Conditions:
A_2 folds



$$A_2^i(t) = \left\{ q \in L \,\middle|\, 1 + \mu_{ti}(q) = 0 \right\}$$

$$A_2^i = \left\{ q \in L \,\middle|\, 1 + \mu_{ti}(q) = 0 \quad \textit{for some} \quad t \right\}$$

Feldbrugge, vdW et al. 2017a

Caustic Conditions:
A_3 cusps

Folding A_2^i manifold into more complex configurations:

For j ≠ i, there is a nonzero tangential vector $\vec{T}$ such that caustic condition

$$\alpha_j = \vec{v}_j^*(q_s)\cdot\vec{T} = 0 \qquad\qquad j \neq i$$

$$\vec{T}(\vec{q}) \parallel \vec{v}_i(\vec{q}) \quad\Rightarrow\quad \vec{v}_i(\vec{q}) \perp \vec{n}(\vec{q}) = \vec{\nabla}\mu_i(\vec{q}) \quad\Rightarrow\quad \mu_{ti,i}(\vec{q}) = \vec{n}\cdot\vec{\nabla}\mu_i = 0$$

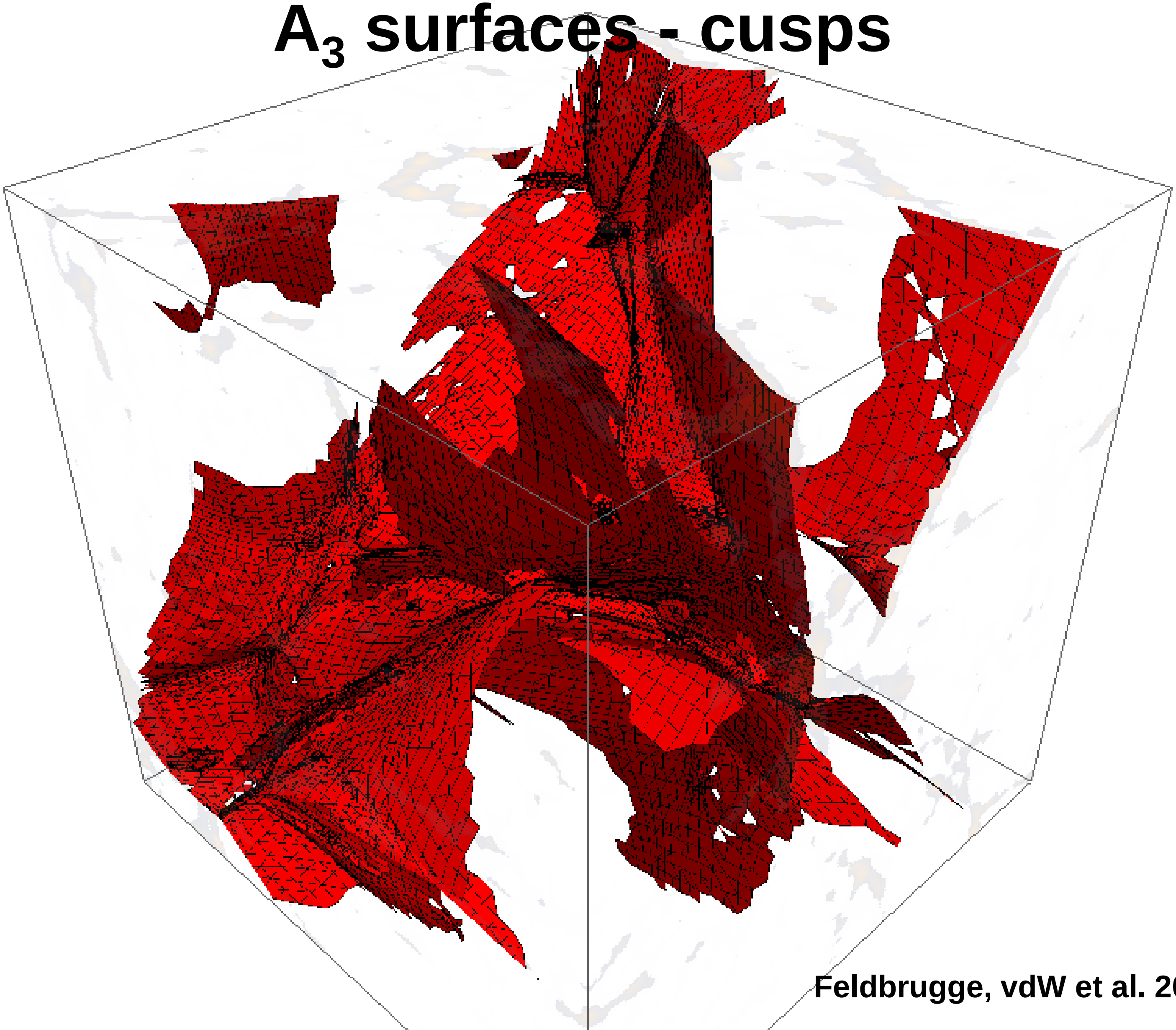
$$A_3^i(t) = \left\{ q \in L \,\middle|\, q \in A_2^i(t) \;\wedge\; 1 + \mu_{ti,i}(q) = 0 \right\}$$

$$A_3^i = \left\{ q \in L \,\middle|\, q \in A_2^i(t) \;\wedge\; 1 + \mu_{ti,i}(q) = 0 \qquad \textit{for some } t \right\}$$

Feldbrugge, vdW et al. 2017a

Skeleton (3D) Cosmic Web:

A_3 surfaces - cusps



Feldbrugge, vdW et al. 2017b

Caustic Conditions: A_4 swallowtails

Folding A_3^i manifold into even more complex configurations:

For j ≠ i, there is a nonzero tangential vector $\vec{T}$ such that caustic condition

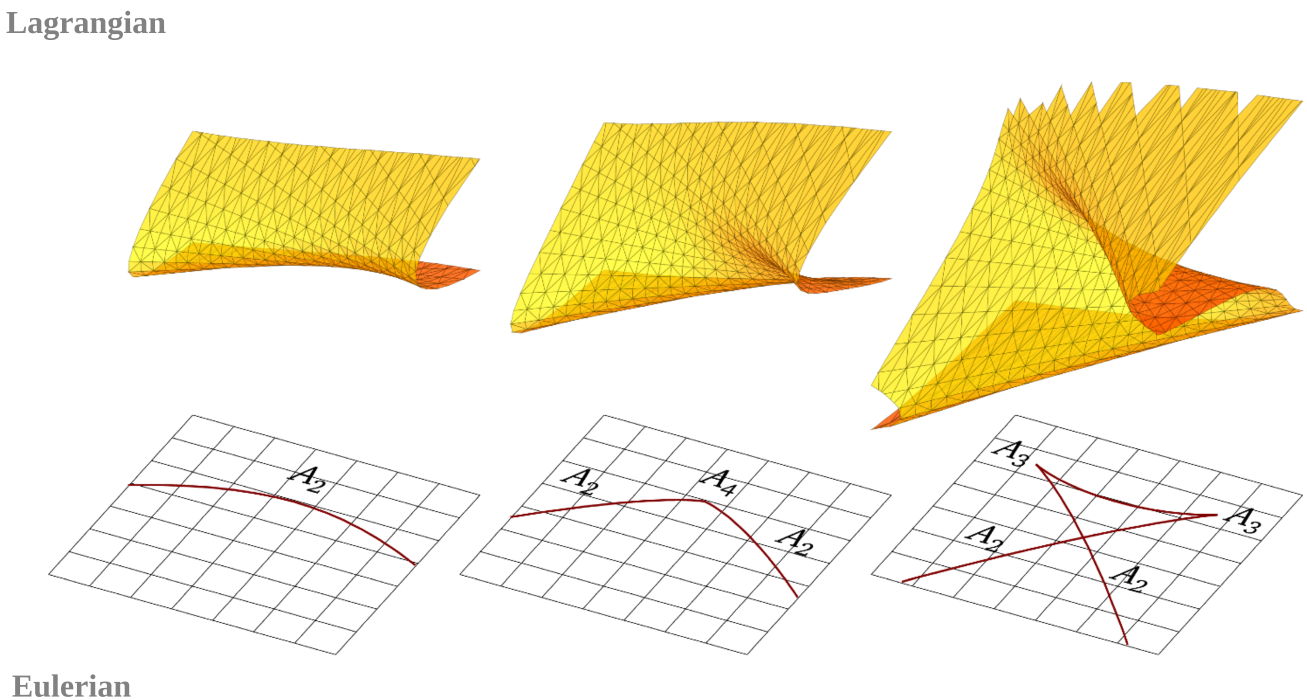
$$\mu_{ti,ii}(\vec{q}) = \vec{v}_i \cdot \vec{\nabla}\mu_{ti,i} = 0$$

$$A_4^i(t) = \left\{ q \in L \middle| q \in A_3^i(t) \ \wedge \ \mu_{ti,ii}(q) = 0 \right\}$$

$$A_4^i = \left\{ q \in L \middle| q \in A_3^i(t) \ \wedge \ \mu_{ti,ii}(q) = 0 \qquad for\ some\ t \right\}$$

Feldbrugge, vdW et al. 2017a

Hidding, Shandarin, vdW 2014



Formation of a A_4 swallowtail (2D)

D=1

A_2: folds
A_3: cusps

D=2

A_2: folds
A_3: cusps
A_4: swallowtail

D_4: umbilics

D=3

A_2: folds
A_3: cusps
A_4: swallowtail
A_5: butterfly

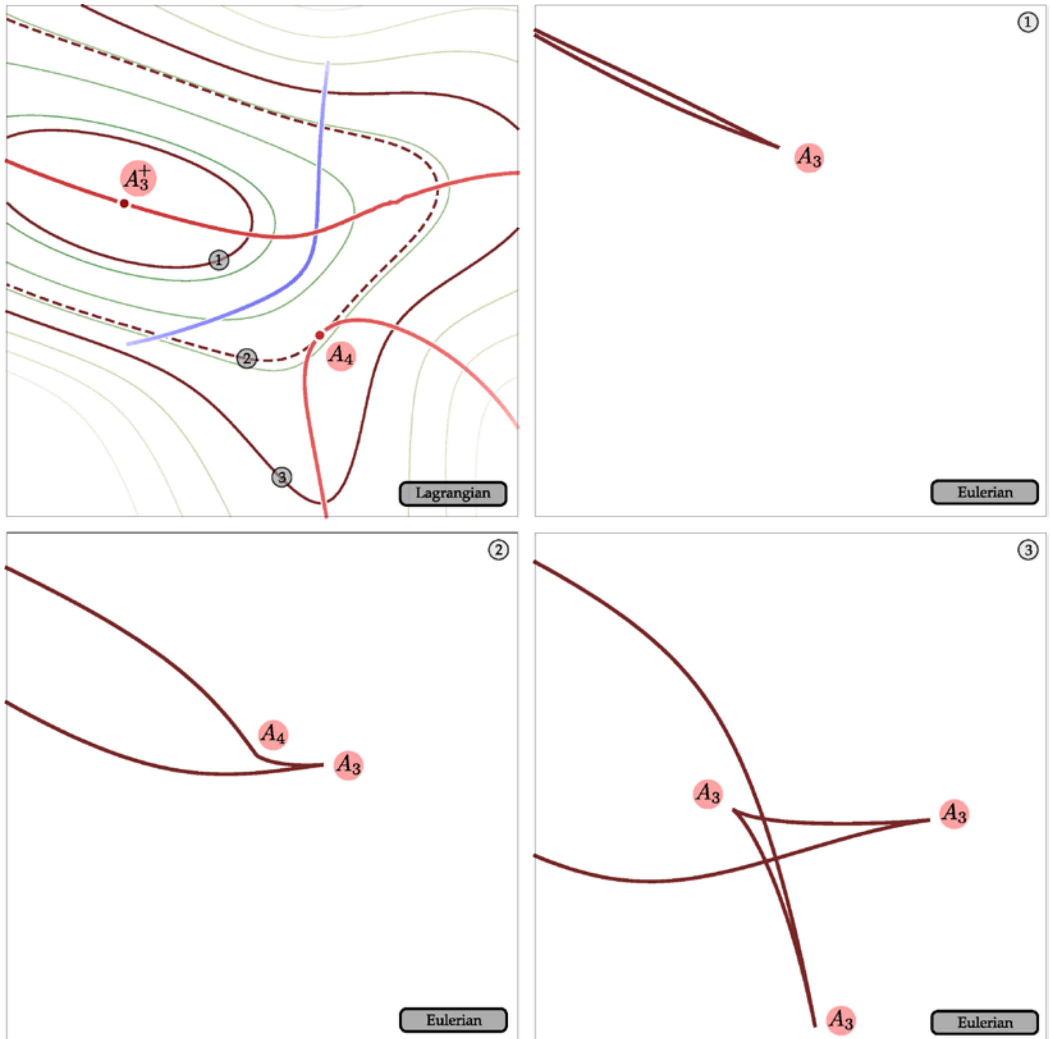
D_4: umbilics
D_5

E_5

Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold
(also see Zeeman, Thom)

Caustic Structures (2D): A_4 swallowtails



Swallowtail catastrophes:

$$\mu_{ti,ii}(\vec{q}) = \vec{v}_i \cdot \vec{\nabla} \mu_{ti,i} = 0$$

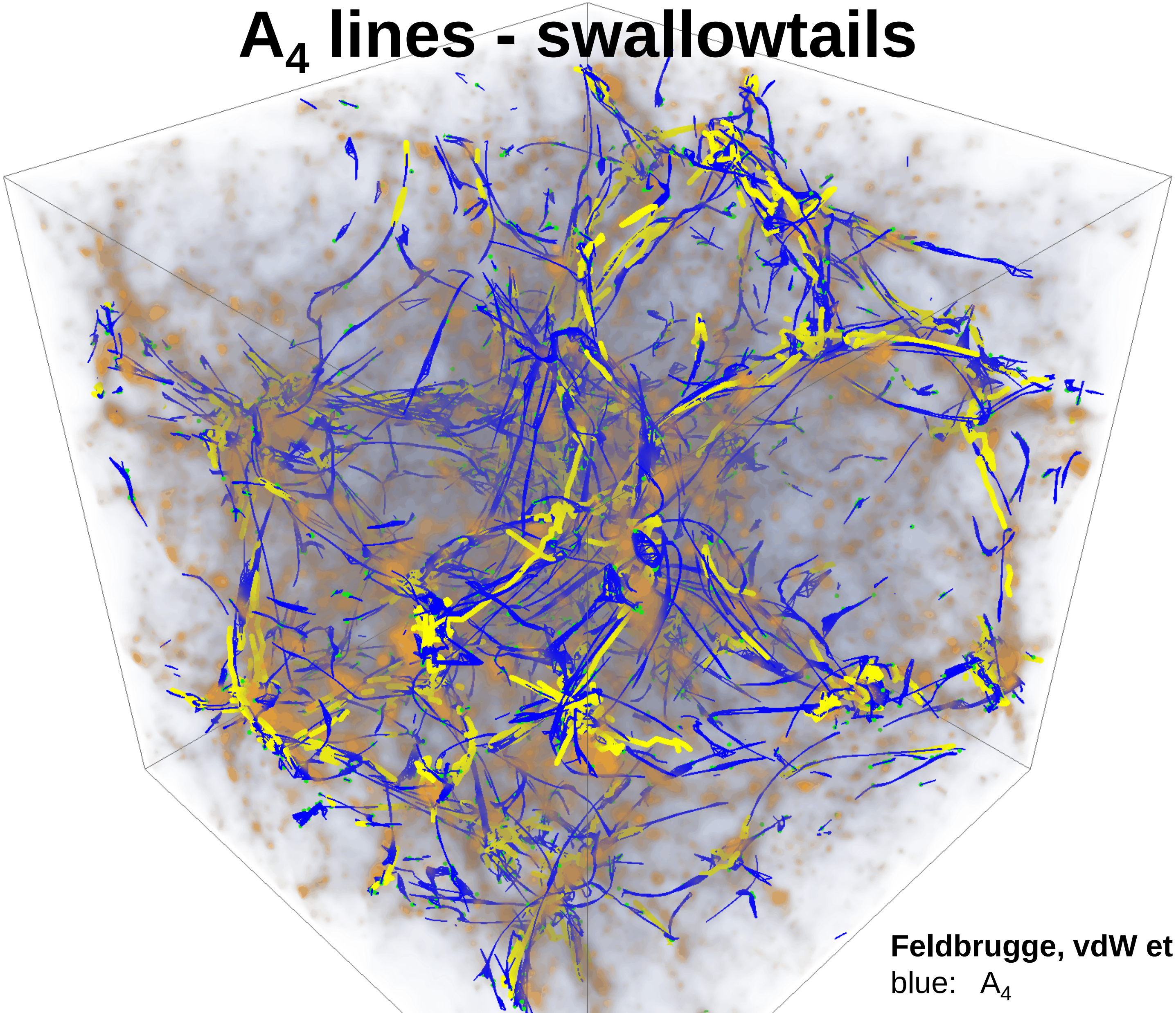
A_4 singularities form there where eigenvector $\vec{v}_i\,(q_s)$ tangential to A_3 line.

Formation of swallowtail singularity.

- Top-left: Lagrangian space.
- Other panels: the formation of a swallowtail singularity in Eulerian space.

Skeleton (3D) Cosmic Web:

A_4 lines - swallowtails



Feldbrugge, vdW et al. 2018

blue: A_4

Umbilics

Umbilic D-class singularities

Configurations determined by 2 eigenvalues:

$$1+\mu_i=0$$
$$1+\mu_j=0$$

$$D_{ij}^4(t)=\left\{\vec{q}\in L \,\middle|\, \vec{q}\in A_2^i(t)\cap A_2^j(t)\right\}$$

In general: isolated singular points (not lines)

2 eigenvalues correlated:
- intersection correlated surfaces highly constrained and complex
- D_4 points are termination points A_3 lines

Lagrangian space:
D_4 singularities and A_3 line connections.

Red dots-circles: D_4^{ij} locations
Black lines: A_3^i lines
Grey lines: A_3^j lines.

2 hyperbolic umbilic singularities
1 elliptical umbilic singularity

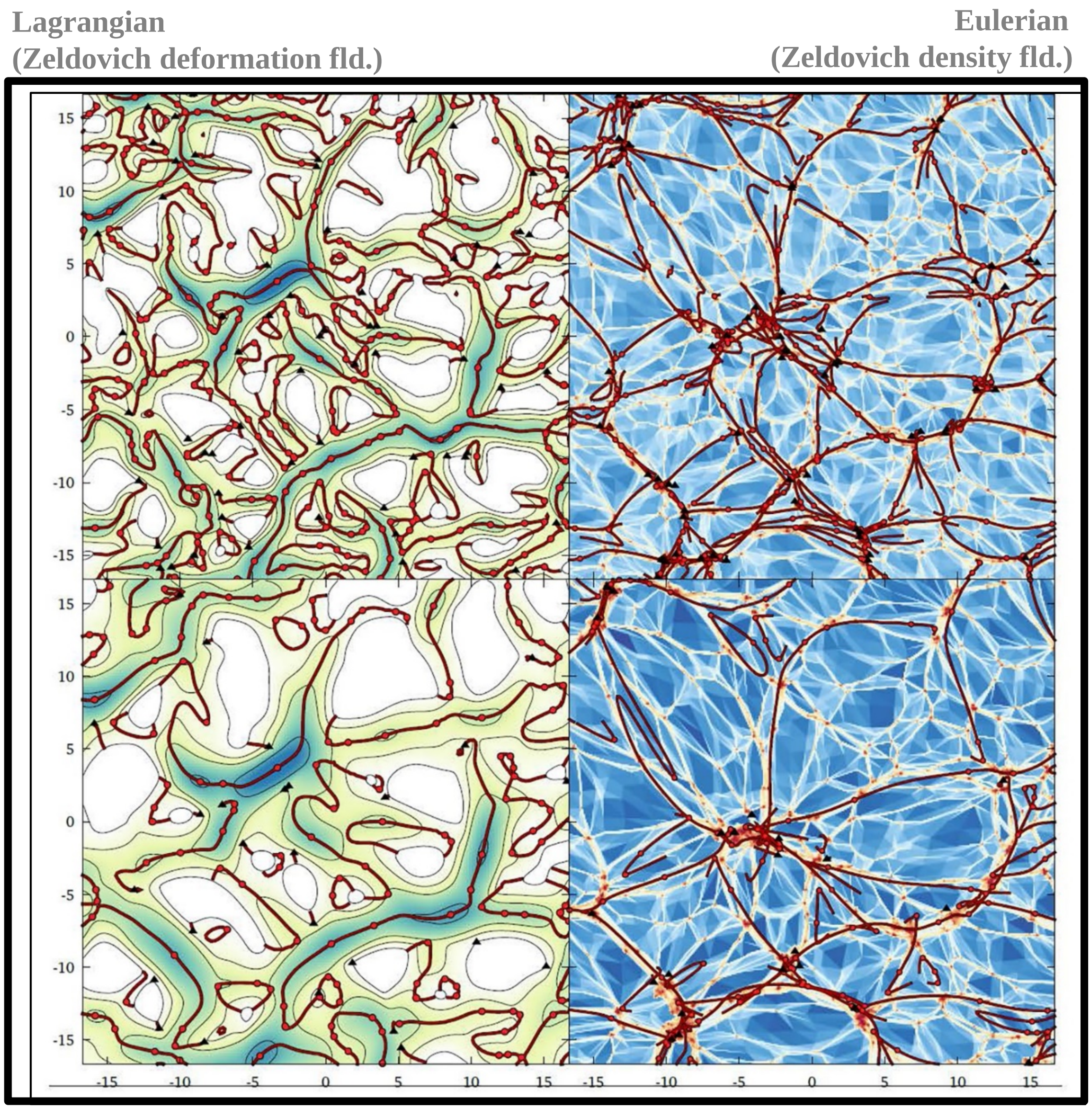
Caustic Skeleton & Cosmic Web

Skeleton (3D) Cosmic Web:
catastrophic connections

Singularity class	Singularity name	Feature in the 2D cosmic web	Feature in the 3D cosmic web
A_2	fold	collapsed region	collapsed region
A_3	cusp	filament	wall or membrane
A_4	swallowtail	cluster or knot	filament
A_5	butterfly	not stable	cluster or knot
D_4	hyperbolic/elliptic	cluster or knot	filament
D_5	parabolic	not stable	cluster or knot

Feldbrugge, vdW et al. 2017b

Skeleton (2D) Cosmic Web:
catastrophic connections



Feldbrugge, vdW et al. 2016

2D Zeldovich density field (log density)

A3 - cusp - red sheets - filaments
A4 - swallowtail - blue lines - nodes

Skeleton (3D) Cosmic Web:

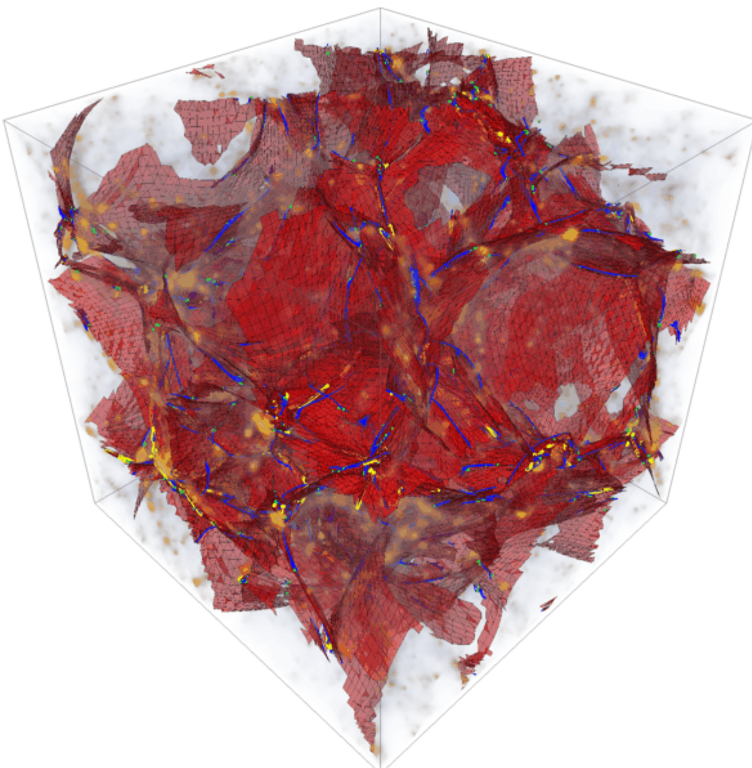
Wall/Membrane formation:

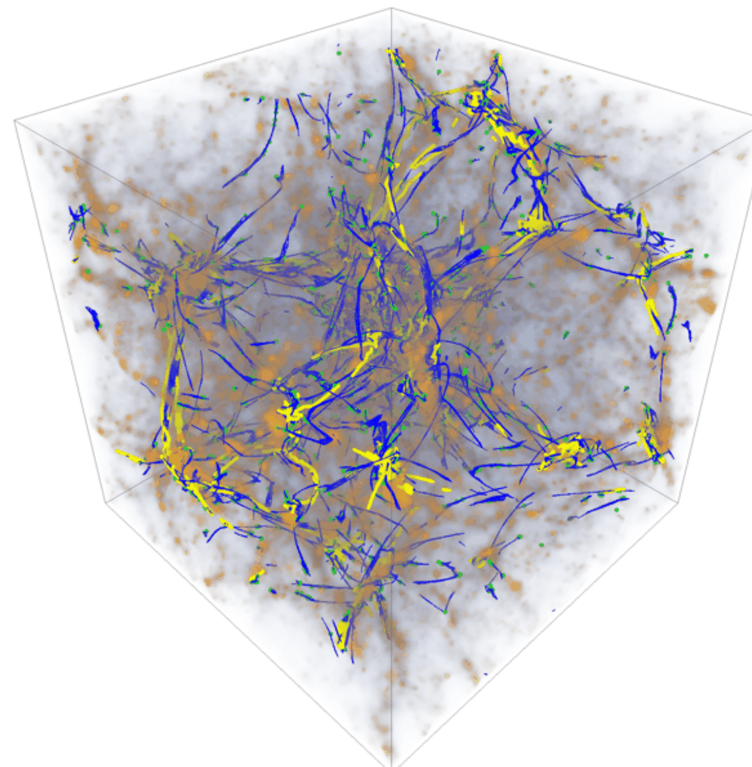
A_2 (cusp) membranes (red):
- collapse along 1 direction

Filament formation:
not necessary to collapse along 2 directions !

A_4 (swallowtail) filaments (blue):
- collapse along 1 direction
- at edges & intersections A_3 sheets

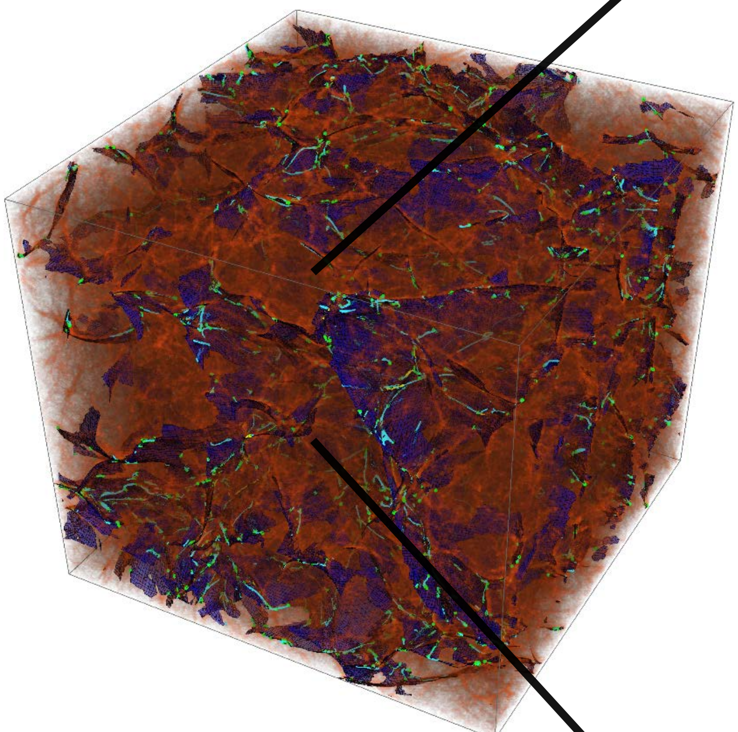
D_4 umbilic filaments (yellow)
- collapse along 2 directions
- higher density filamentary extensions nodes

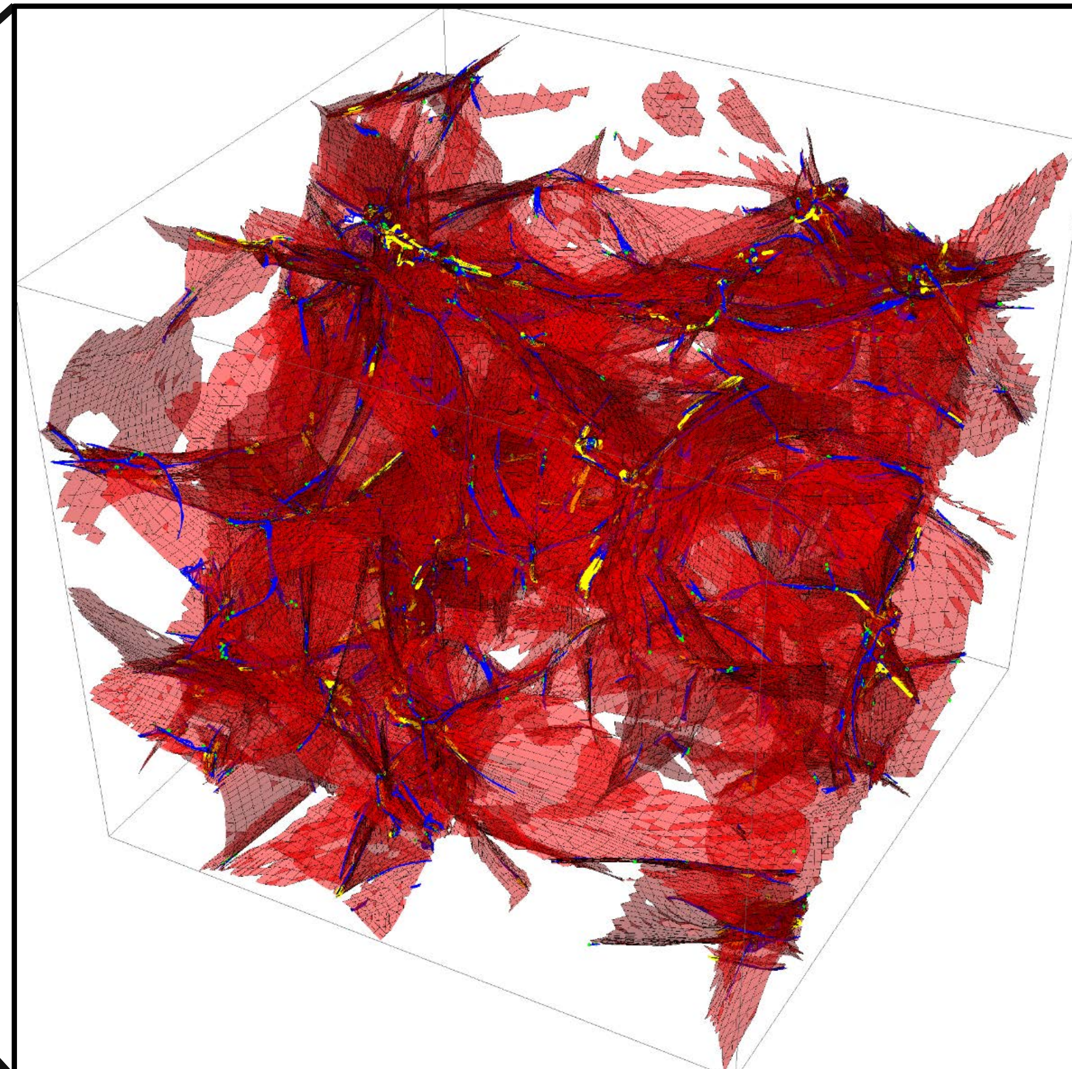




Feldbrugge, vdW et al. 2018

Skeleton (3D) Cosmic Web:
catastrophic connections



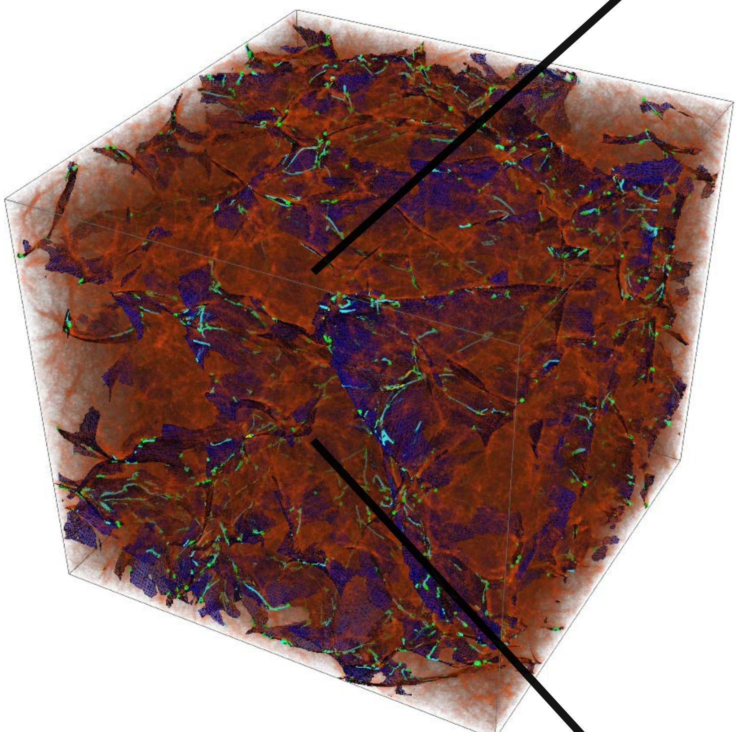


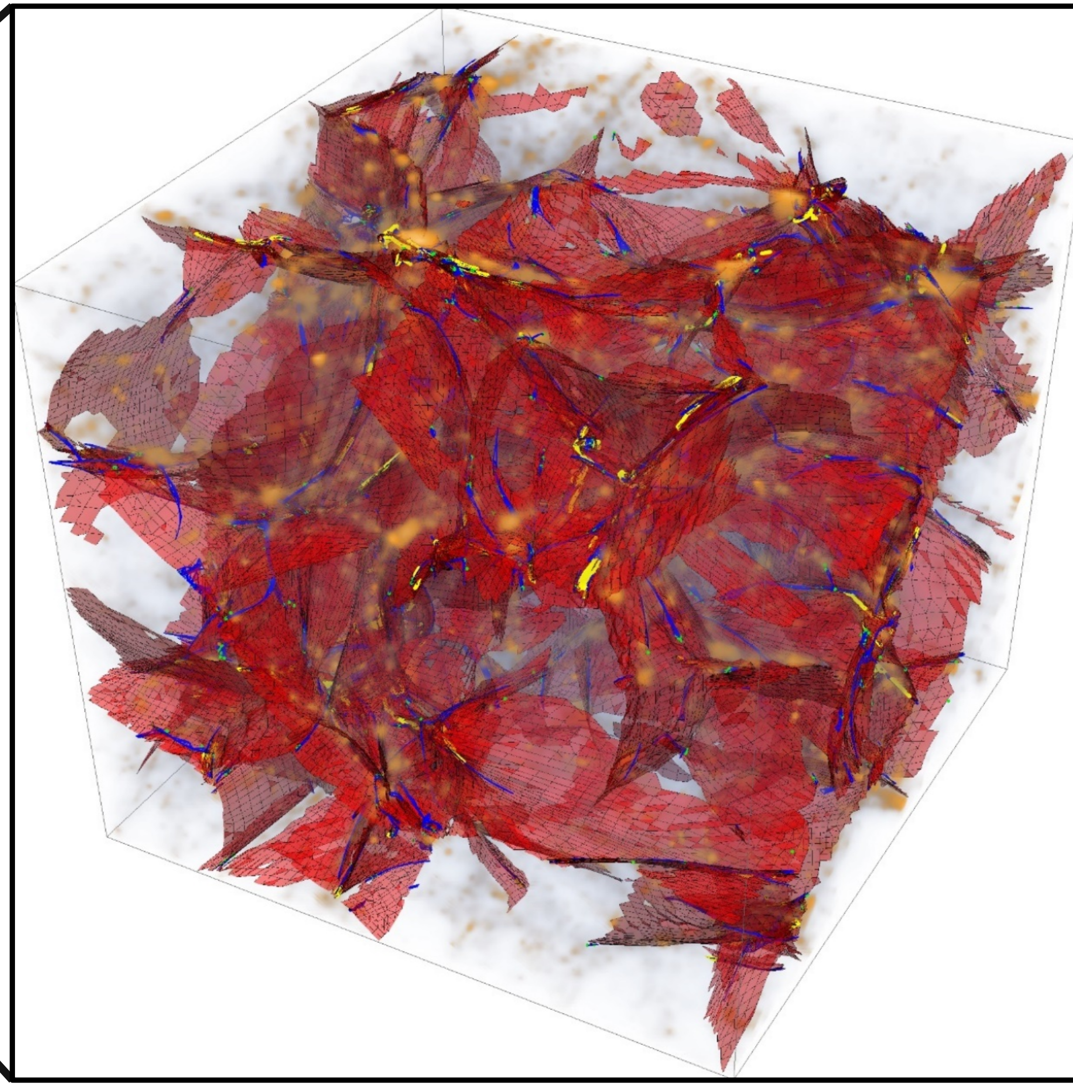
DM Nbody density field (log density)

A3 - cusp - red sheets - walls/membranes
A4 - swallowtail - blue lines - filaments
A5 - butterfly - green points - cluster nodes

Feldbrugge, vdW et al. 2018

Skeleton (3D) Cosmic Web:
catastrophic connections





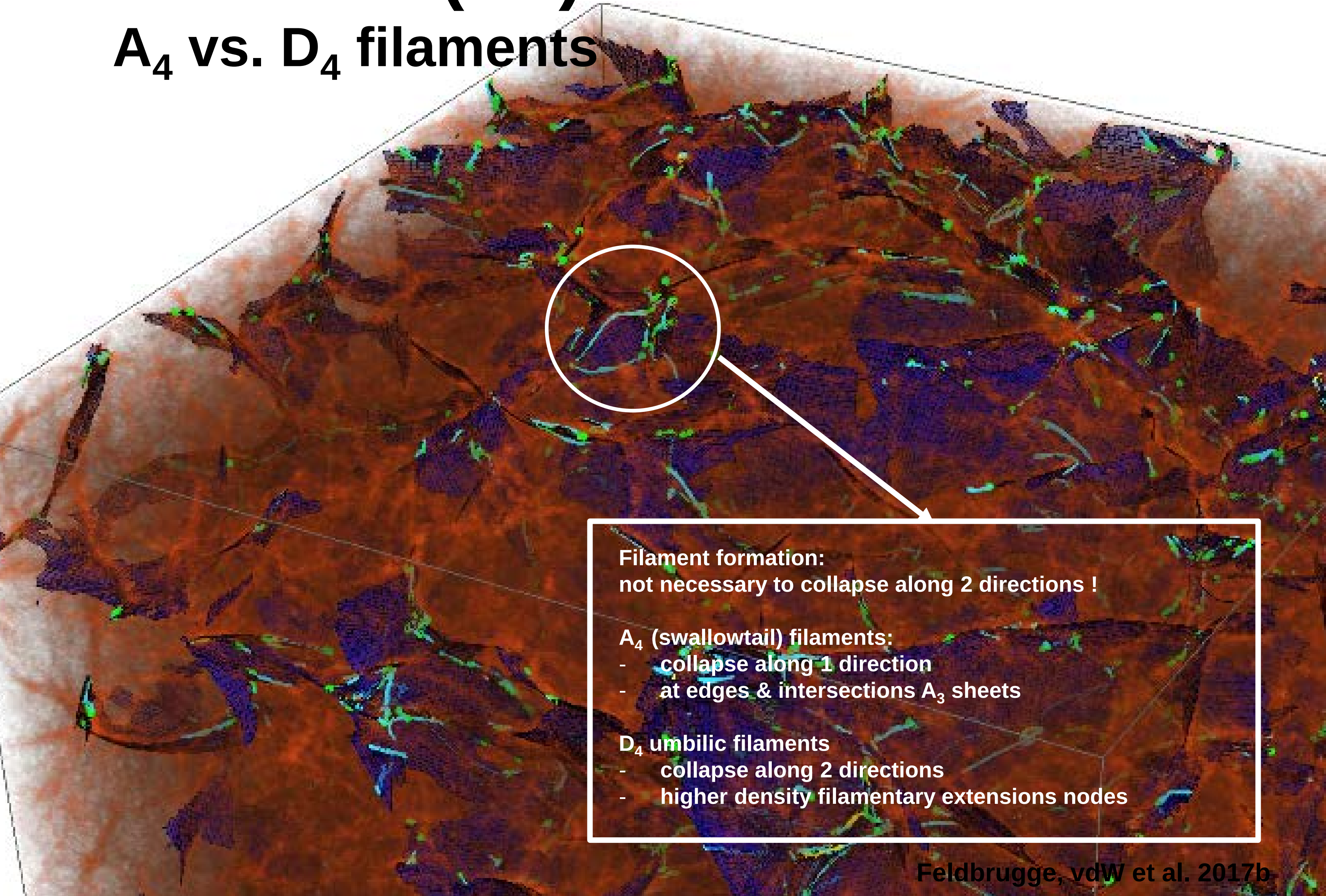
DM Nbody density field (log density)

A3	-	cusp	- red sheets	- walls/membranes
A4	-	swallowtail	- blue lines	- filaments
A5	-	butterfly	- green points	- cluster nodes

Feldbrugge, vdW et al. 2018

Skeleton (3D) Cosmic Web:

A_4 vs. D_4 filaments



Filament formation:
not necessary to collapse along 2 directions !

A_4 (swallowtail) filaments:

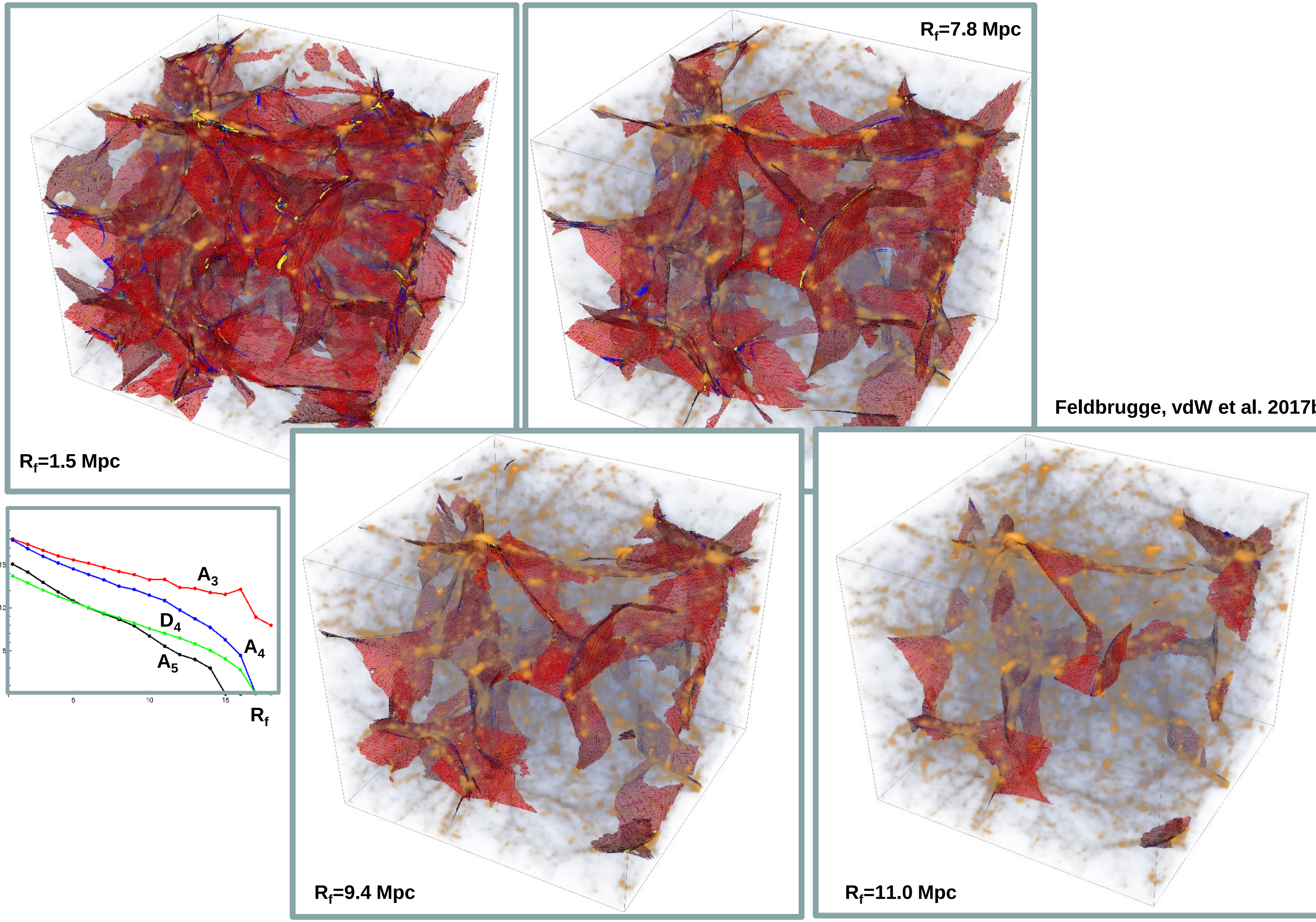
- collapse along 1 direction
- at edges & intersections A_3 sheets

D_4 umbilic filaments

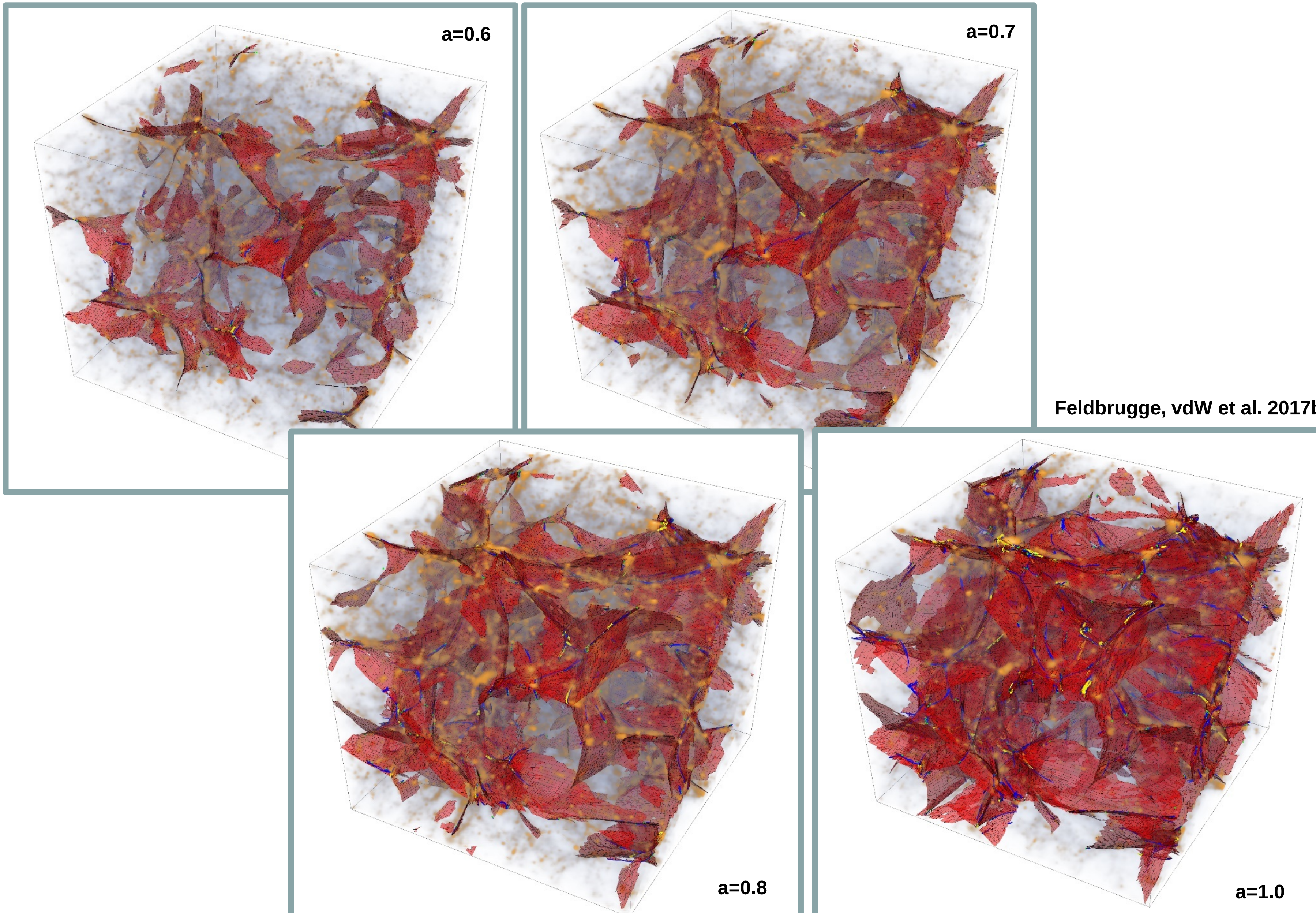
- collapse along 2 directions
- higher density filamentary extensions nodes

Feldbrugge, vdW et al. 2017b

Cosmic Web - Multiscale Skeleton



Cosmic Web - Evolving Skeleton

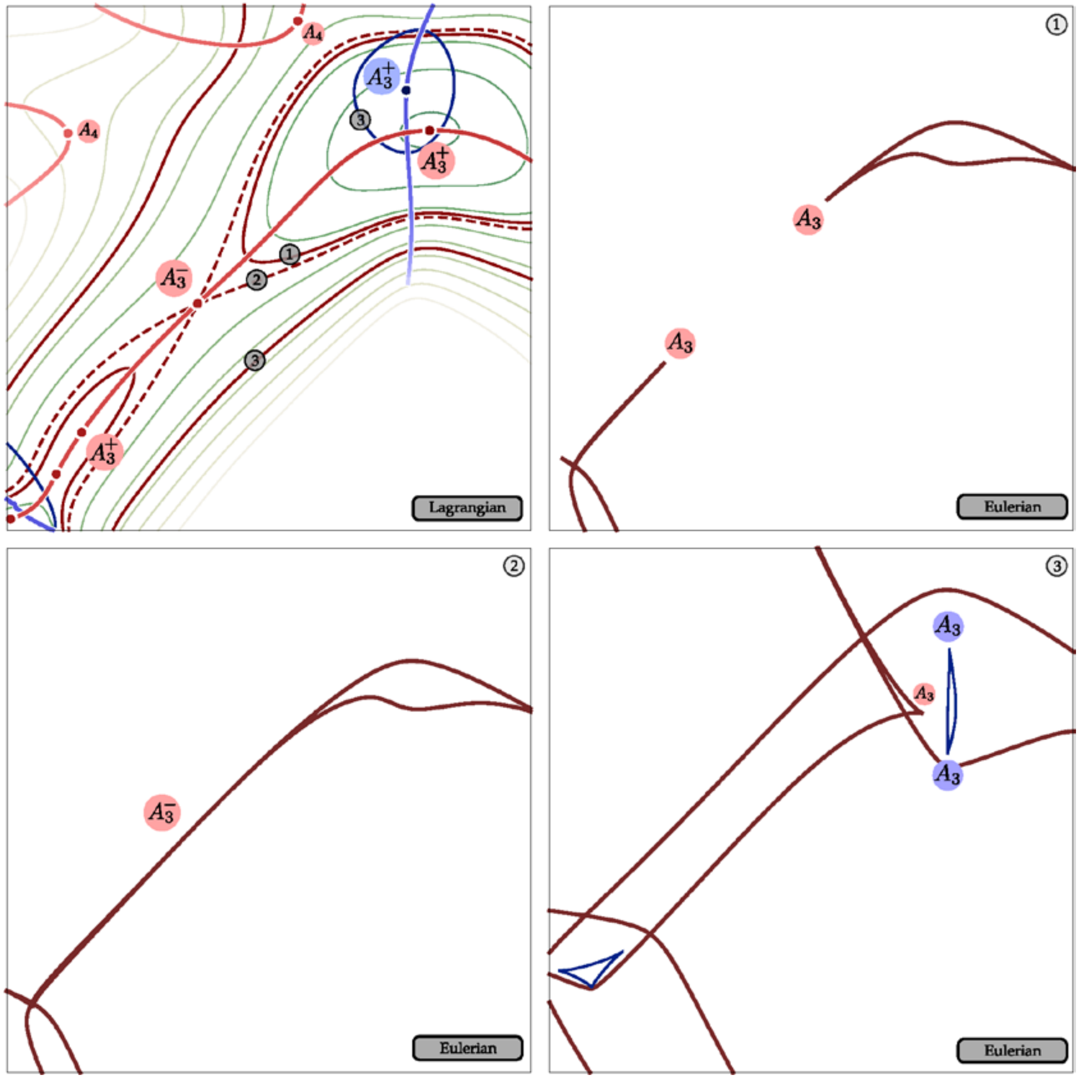


Feldbrugge, vdW et al. 2017b

Caustic Merging & Annihilation

Caustic Hierarchy

Merging & Annihilation of Structures



Evolution & Merging Caustic Features:

Topological transformations
in the deformation field:

- Growth of islands ("pancakes") starts at **maxima (A_3^+)** in deformation field λ_i
- Merging of islands at **saddle points (A_3^-)** in deformation field λ_i

The merger of two pancakes.

- Top-left: Lagrangian space.

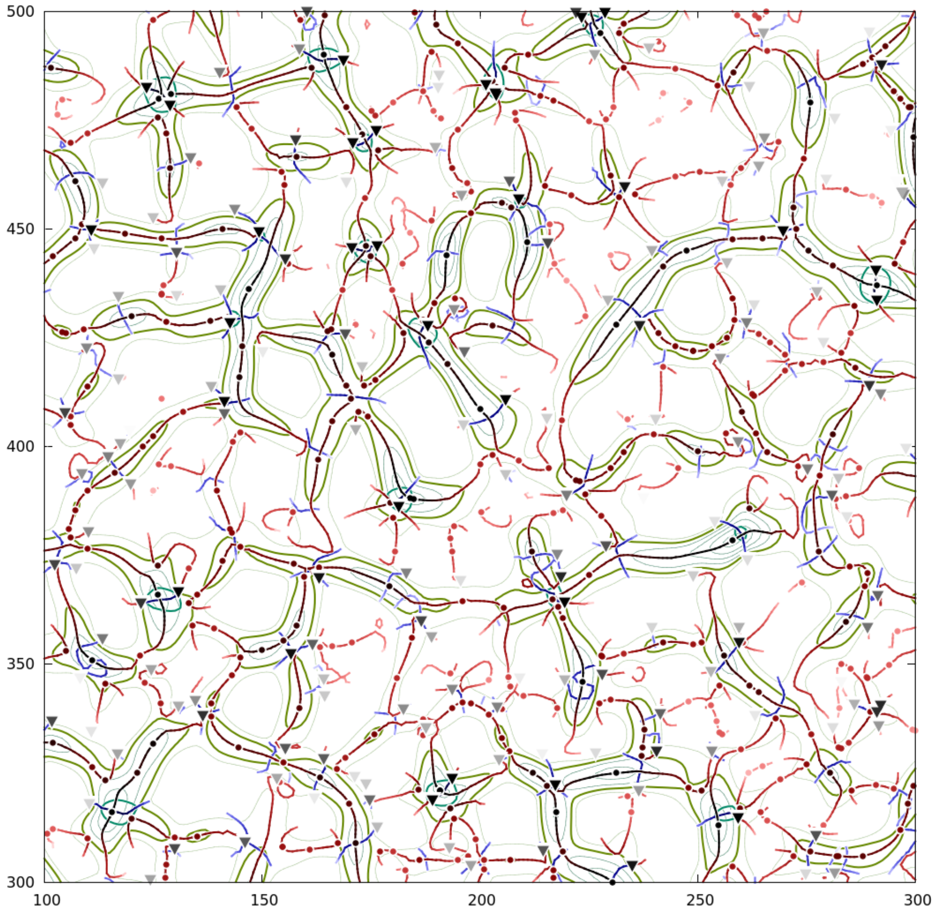
- Other panels: the evolution of caustics in Eulerian space.

Hierarchical Buildup of Cosmic Skeleton:

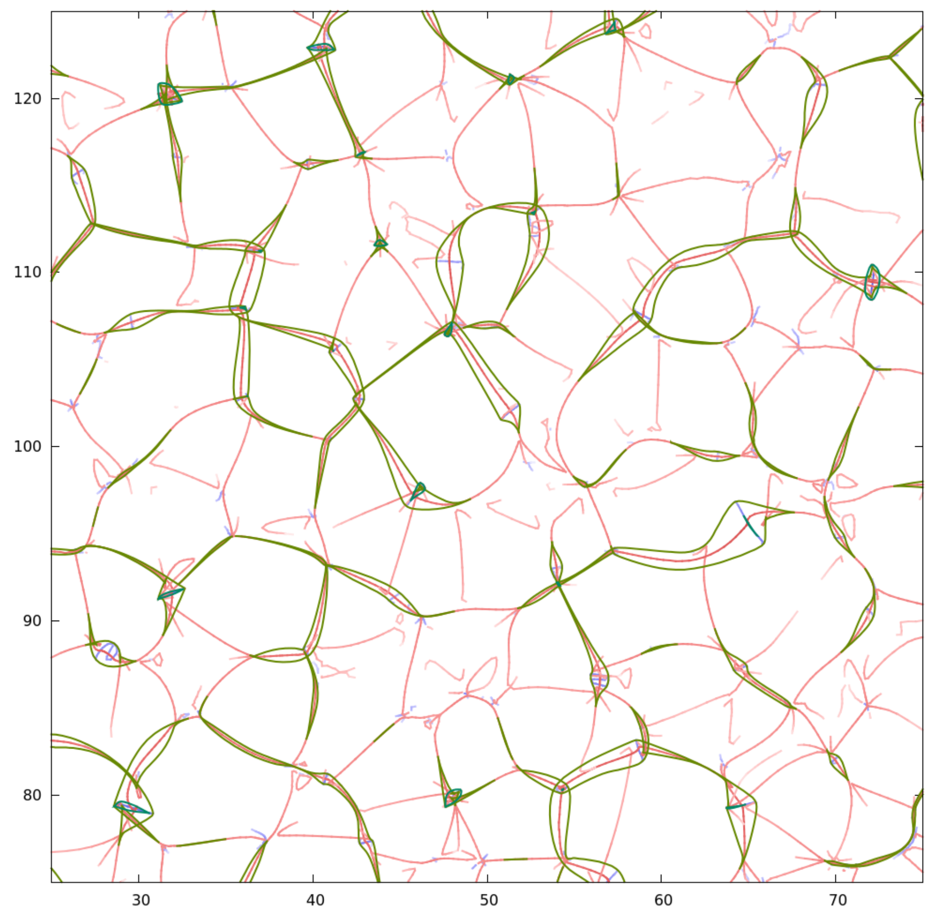
- Topological transformations at maxima and saddles
- Persistence diagrams (birth-death)

Caustic Hierarchy

Merging & Annihilation of Structures



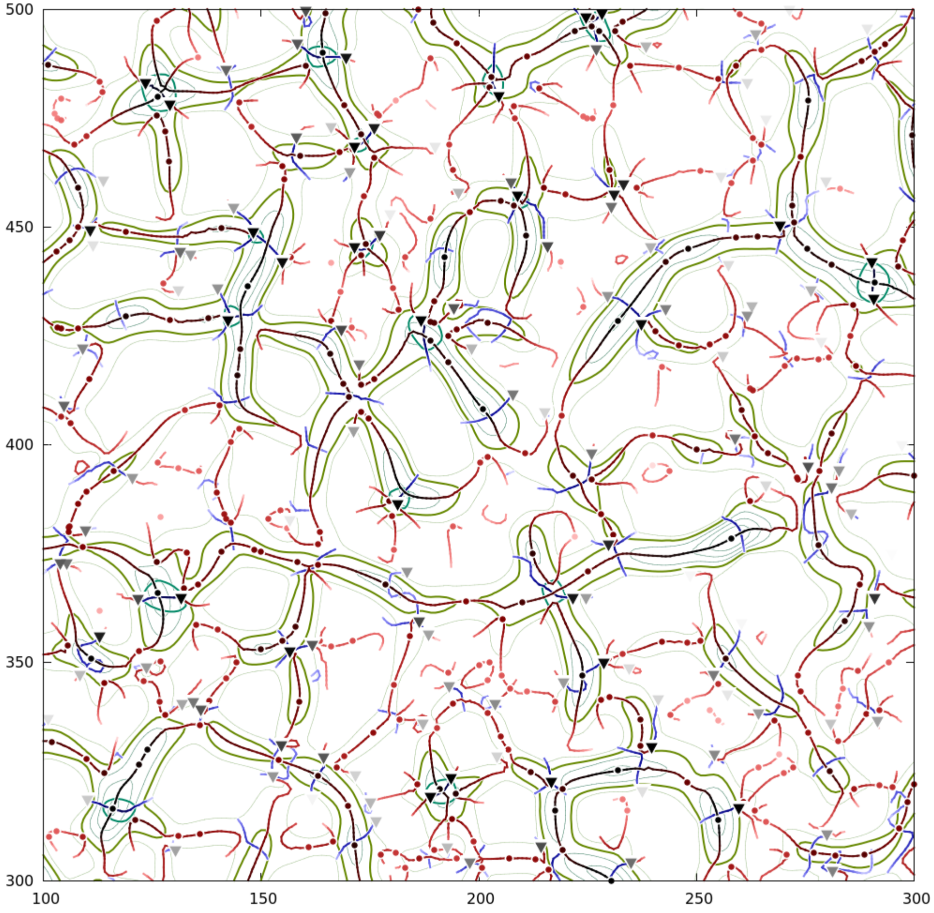
Lagrangian



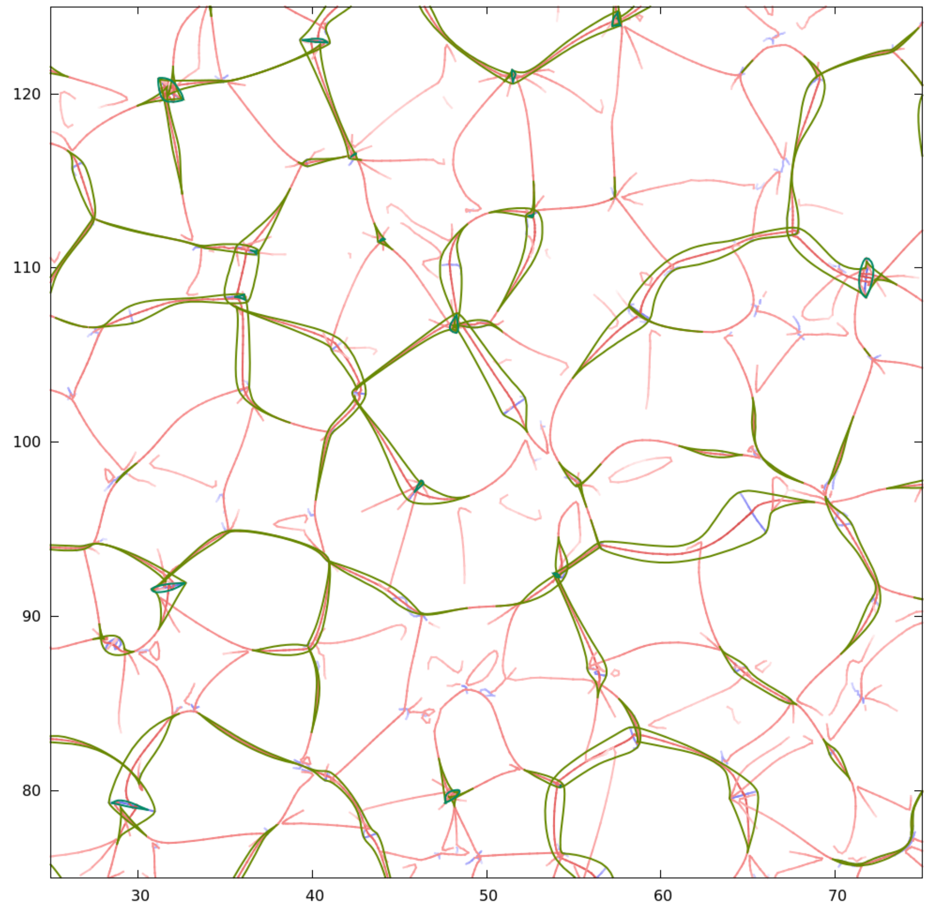
Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



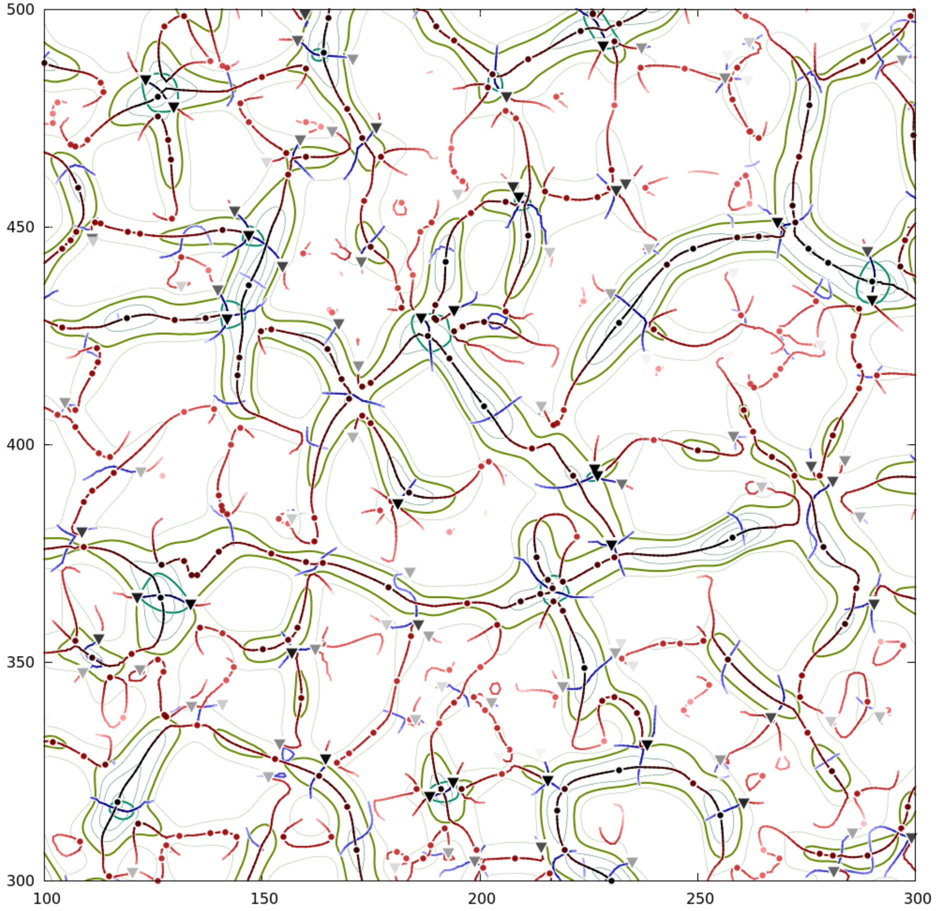
Lagrangian



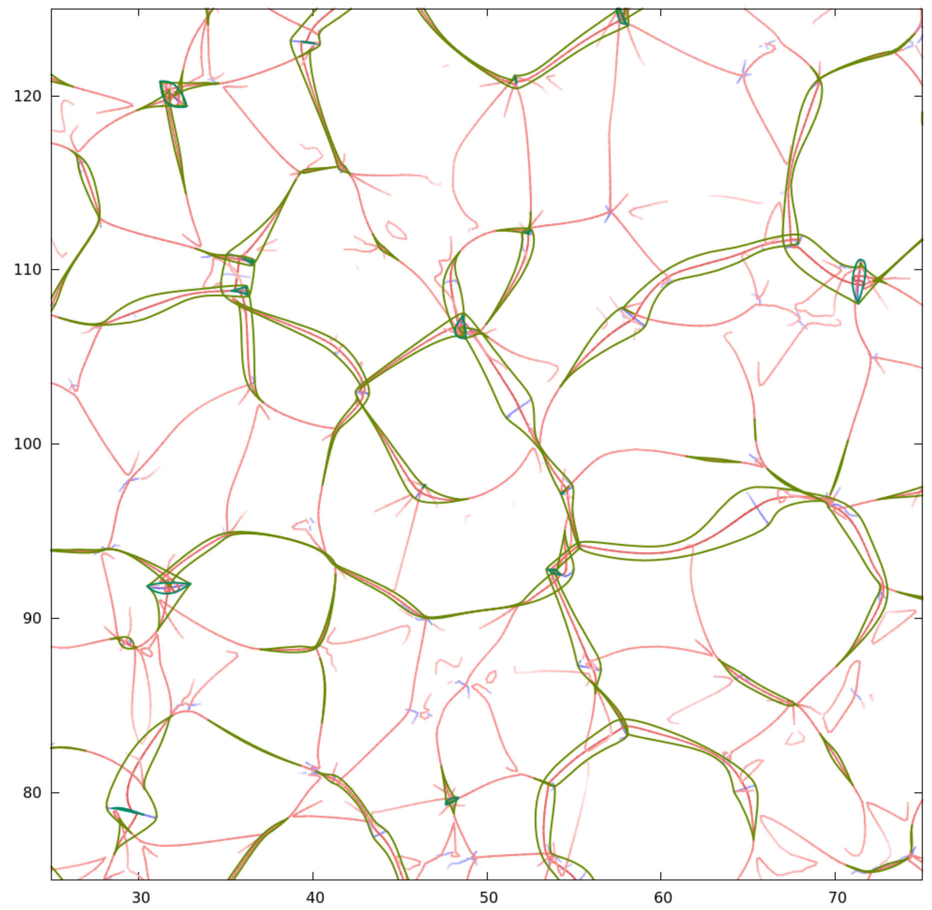
Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



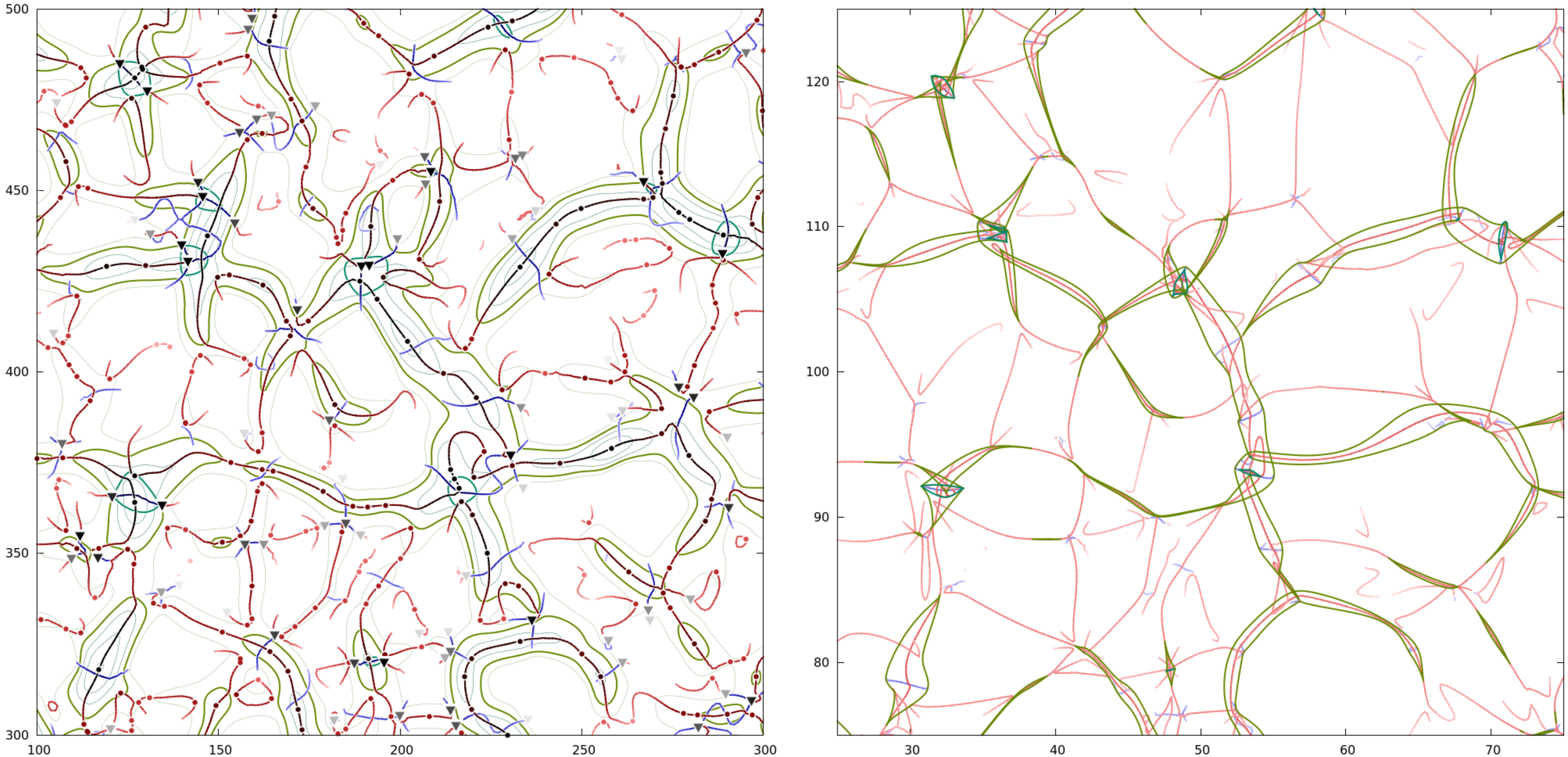
Lagrangian



Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures

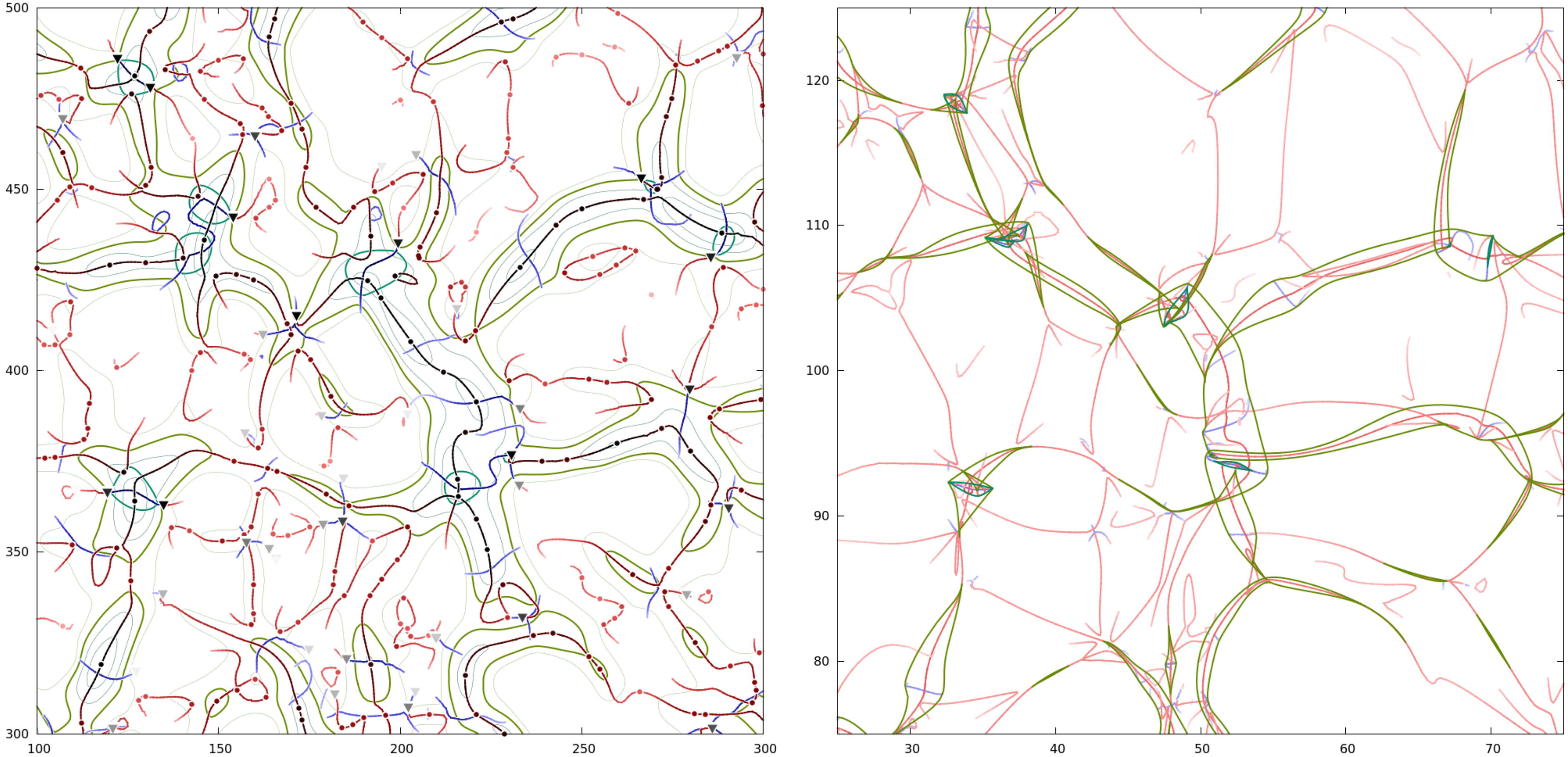


Lagrangian

Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



Lagrangian

Eulerian

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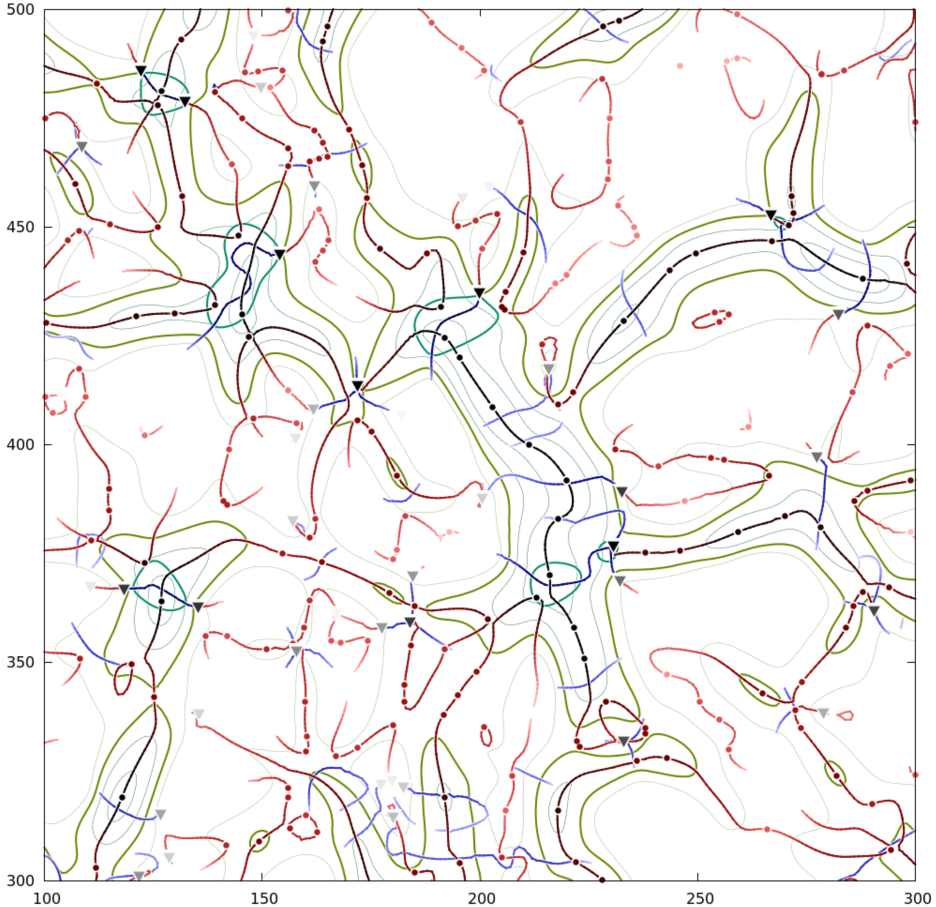
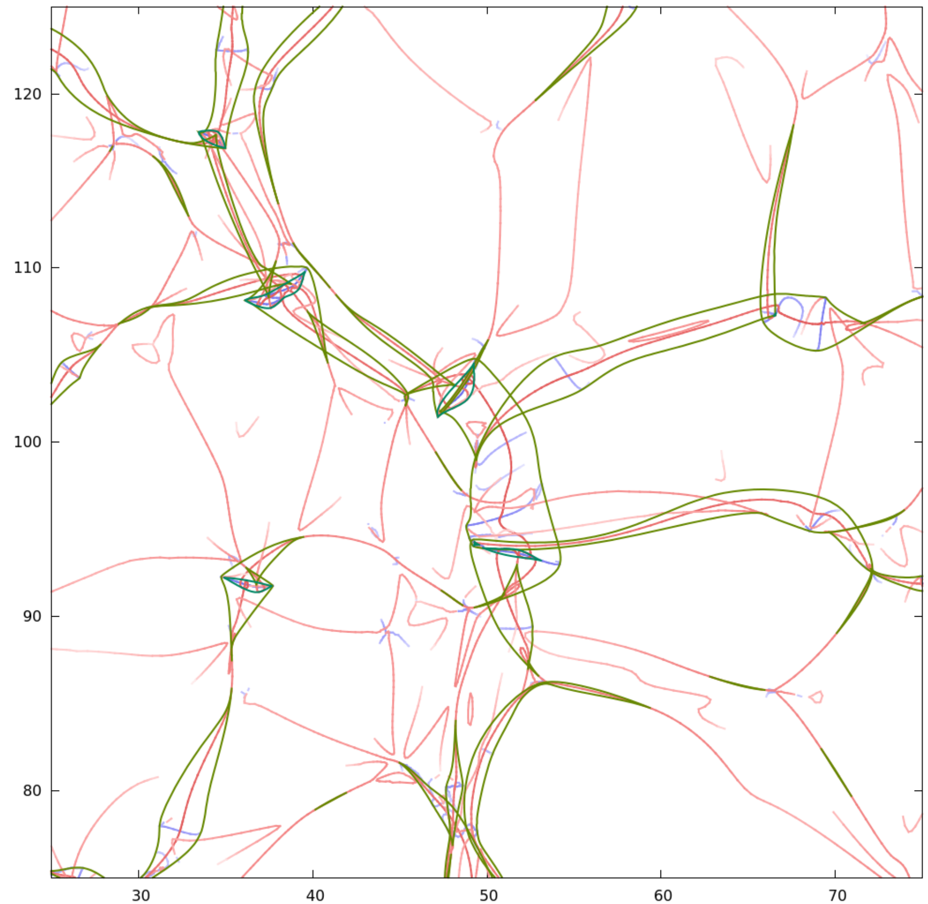
Merging & Annihilation of Structures



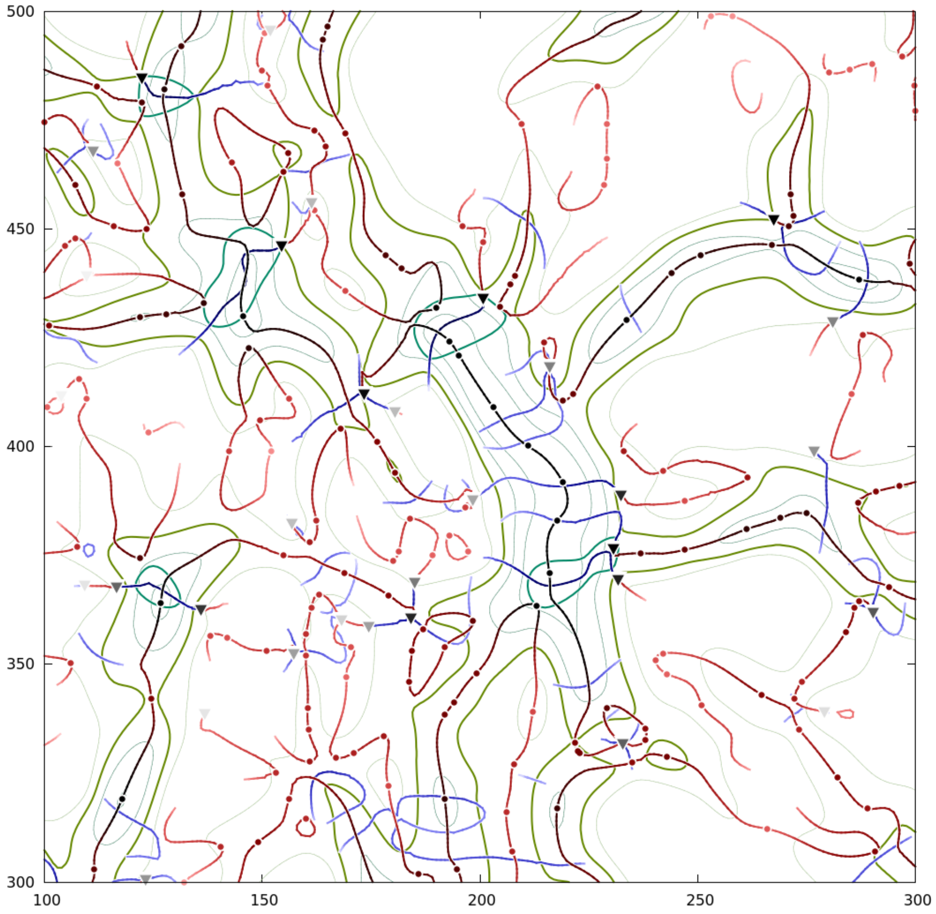


Lagrangian

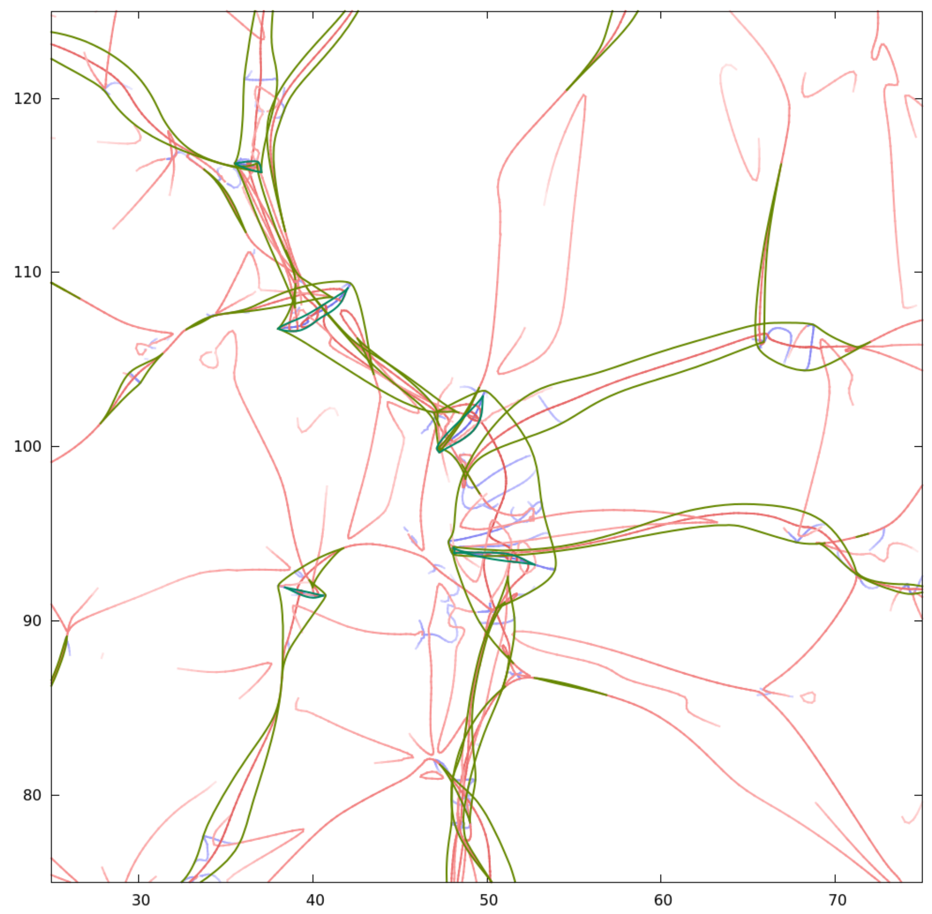
Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



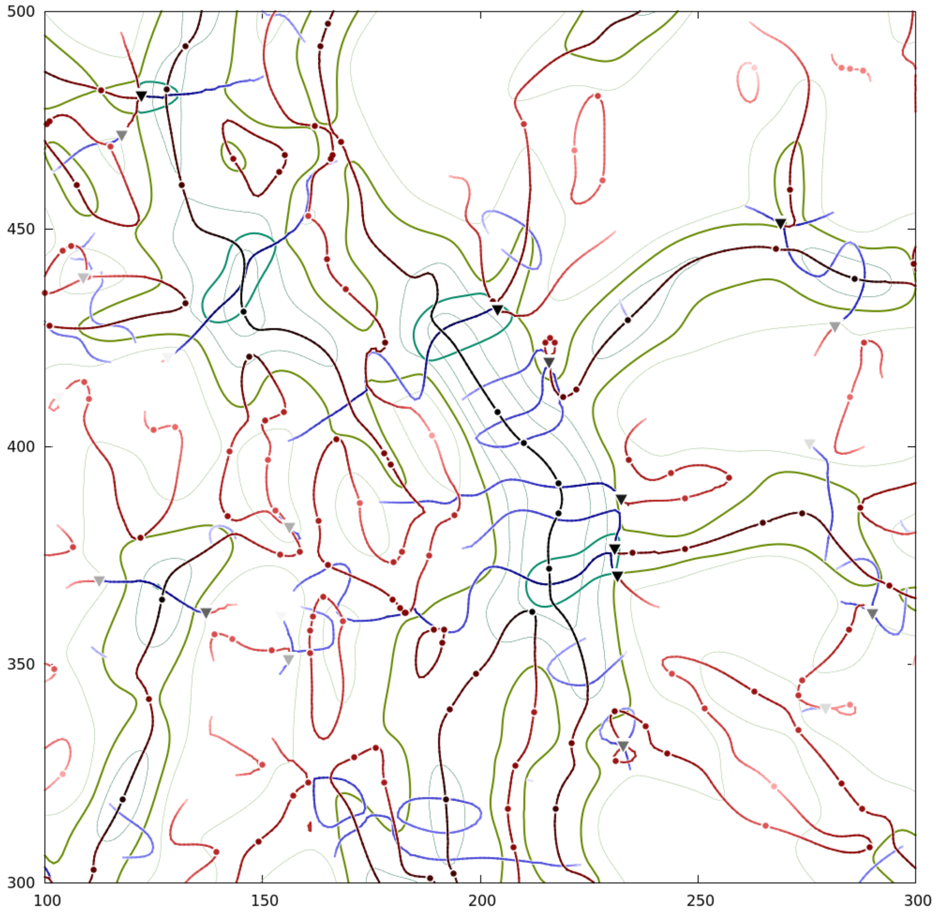
Lagrangian



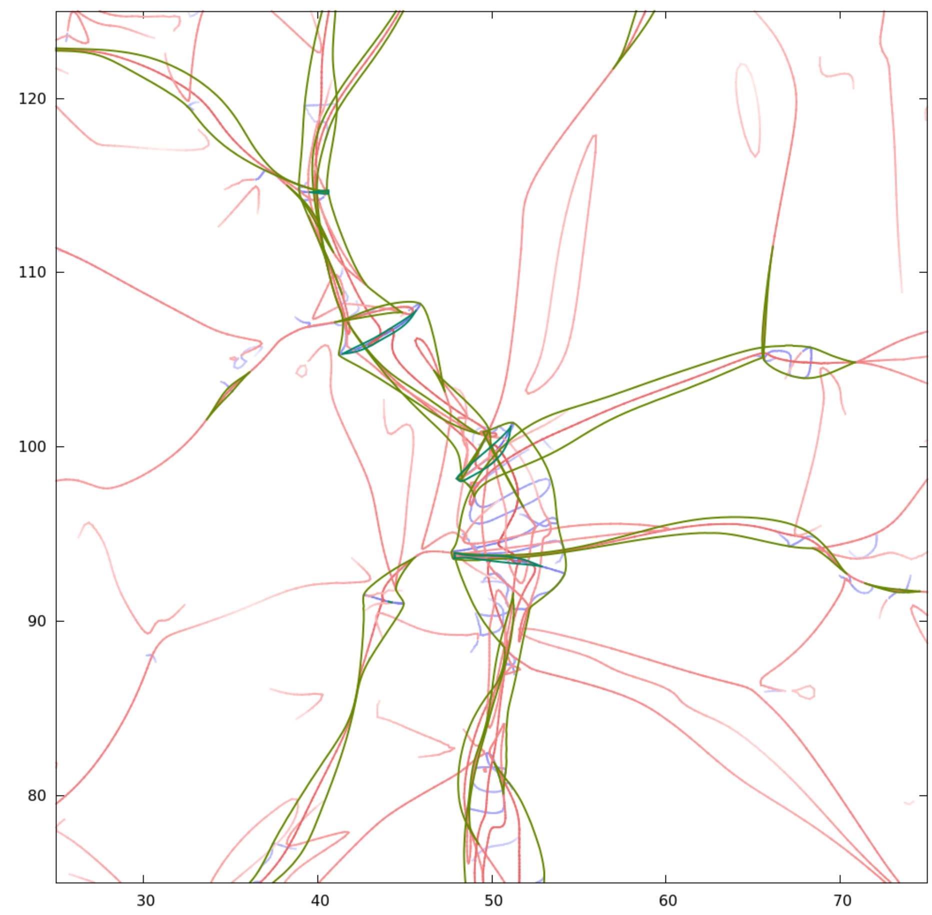
Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



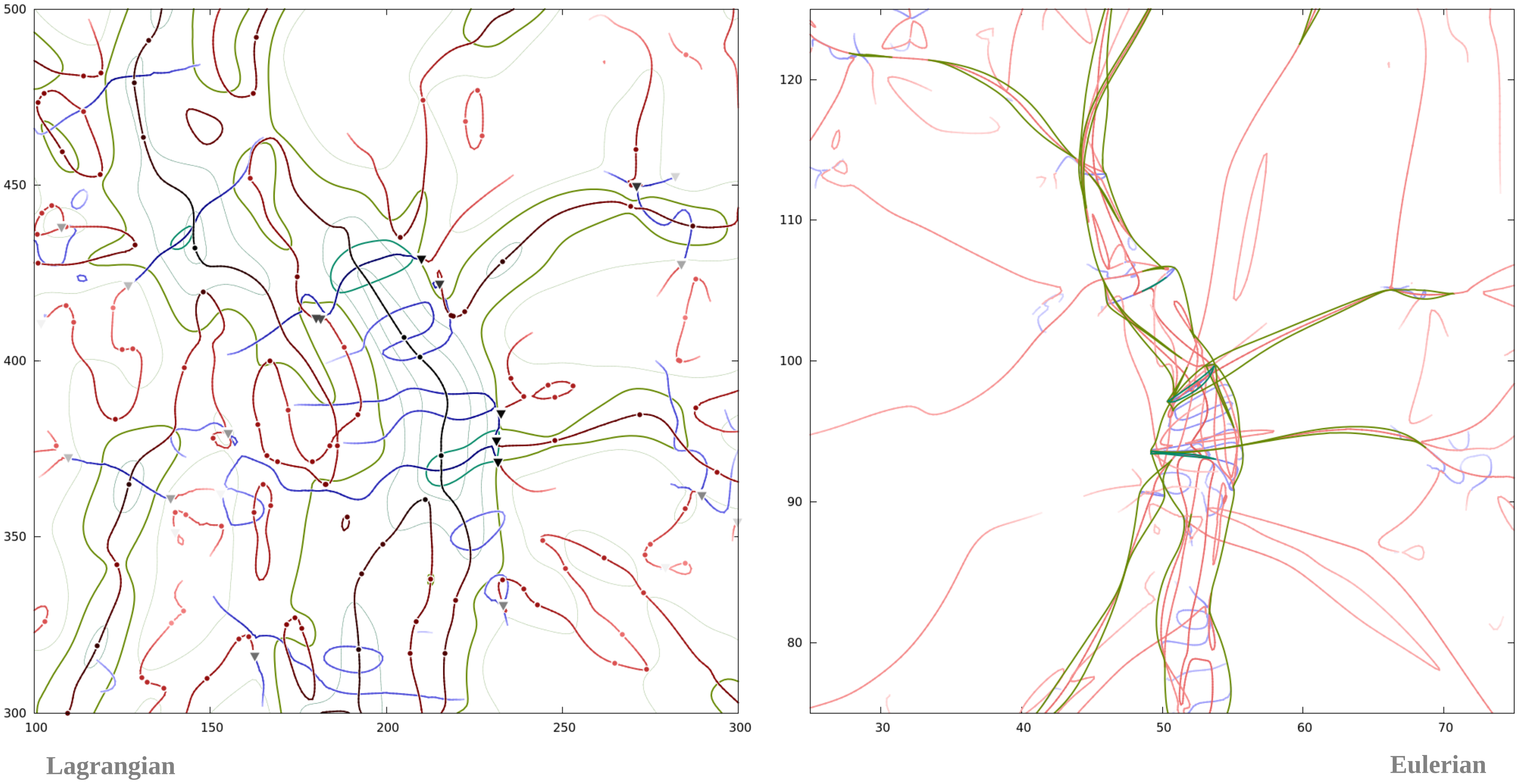
Lagrangian



Eulerian

Caustic Hierarchy

Merging & Annihilation of Structures



Caustic Web:

Connectivity
&
Persistent Topology

Topology & Morse

Relation to Morse Theory:

Topological Structure Continuous Field

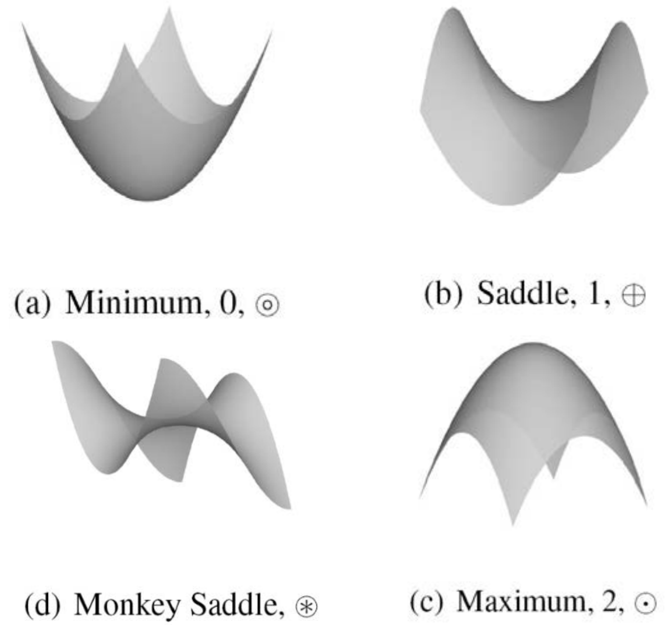
determined by **singularities:**

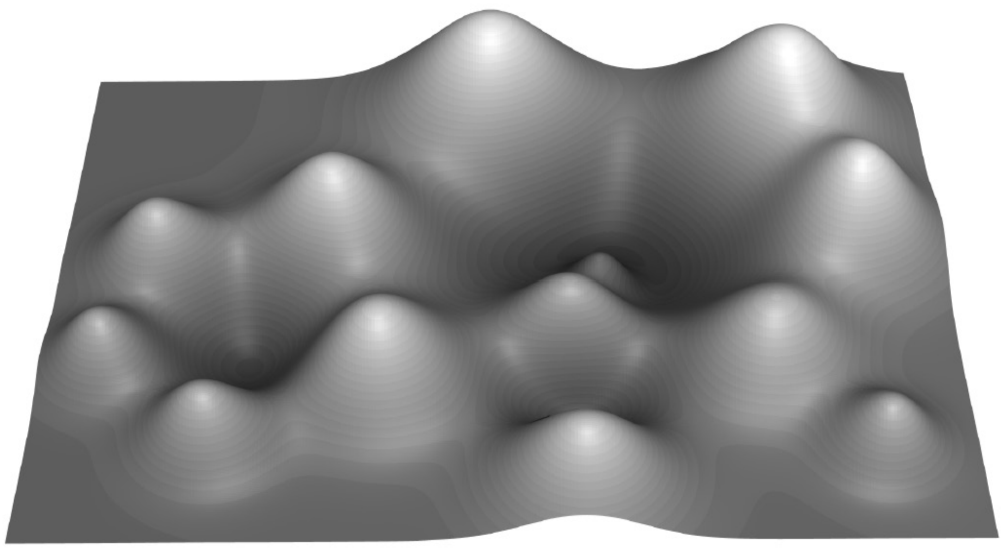
ζ_0 : **minima**

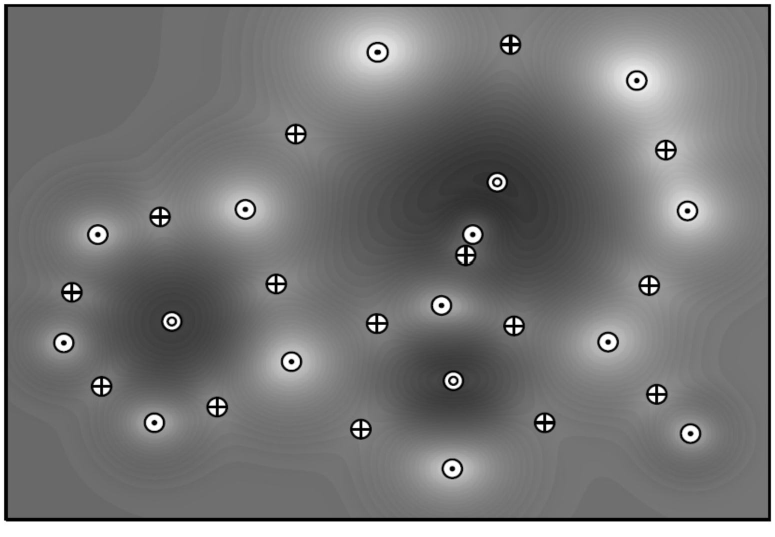
ζ_1 : **saddle 1**

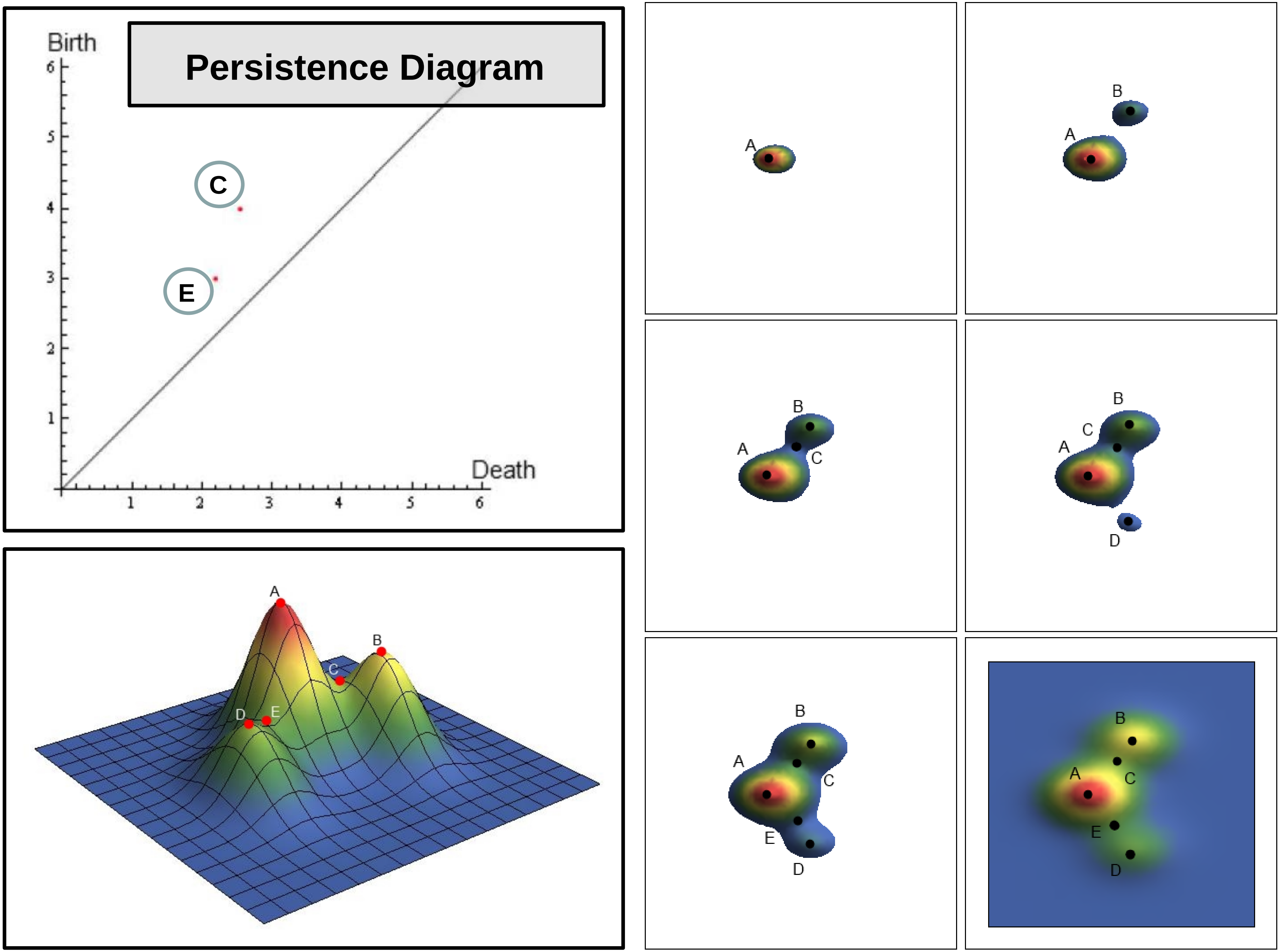
ζ_2 : **saddle 2**

ζ_3 : **maxima**









Persistence Diagram
Birth
Death
1
2
3
4
5
6
C
E
A
B
C
D
E

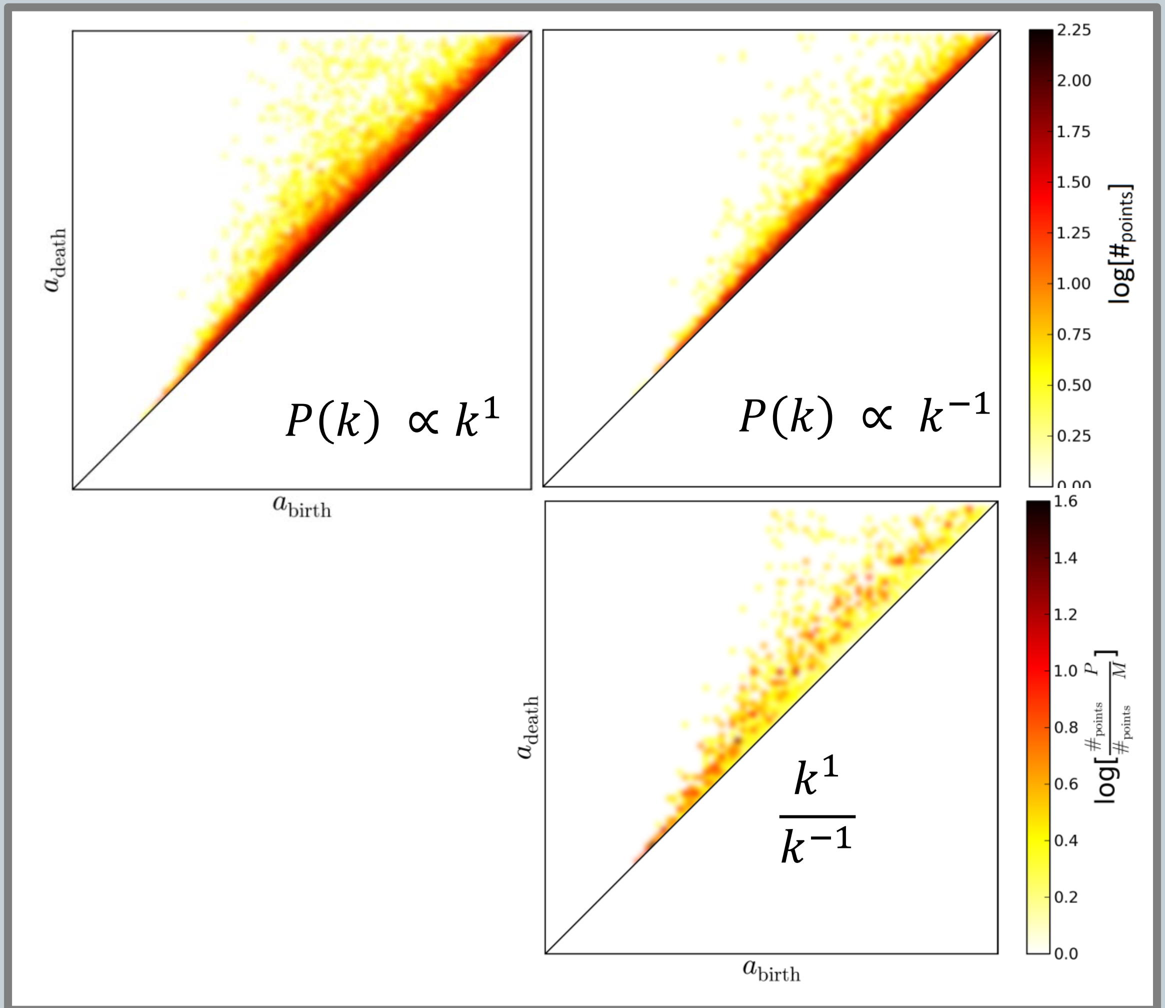
Persistence and Merger Trees

- Merger tree is only based on one parent object!
- Combine information of all merger trees into

Persistence Diagram

(Edelsbrunner et al. 2000)

- Information w.r.t. formation and disappearance of structures due to hierarchical evolution
- Not only mathematical principle.



Caustic Skeleton Summary

- Full phase-space dynamics elucidates the intimate link between multistream flow field and morphological identity of structures.

- 6D Phase-space wrapping of 3D dark matter sheet leads to the emergence of caustic singularities. Their connection establishes the skeleton of the cosmic web.

- Full analytical formalism for caustics in 3-D (and any dimensional) space

- To outline the skeleton of the resulting caustic skeleton, not only
  EIGENVALUES but also the EIGENVECTORS
  of fundamental importance

- Filaments in 2 different species:
  A_4 swallowtail filaments: collapse along 1 direction (boundaries of membranes)
  D_4 umbilic filaments: collapse 2 directions