#### **Topology of the cosmic web**



#### Sandrine Codis - Institut d'Astrophysique de Paris -



#### **Topology of the cosmic web**



# How is the cosmic web woven?



# Vlasov-Poisson equations: dynamics of a self-gravitating collisionless fluid

Liouville theorem:

$$\begin{bmatrix} \frac{\partial}{\partial t} + \frac{\mathbf{p}}{ma^2} \frac{\partial}{\partial \mathbf{x}} - m \nabla \phi \frac{\partial}{\partial \mathbf{p}} \end{bmatrix} f(\mathbf{x}, \mathbf{p}, \mathbf{t}) = \mathbf{0}$$
$$\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$$

**Poisson equation:** 

These highly non-linear equations can be solved using numerical simulations or analytically in some specific regimes. Exact solutions are crucial to understand the details of structure formation.

Before shell-crossing, moments>2 can be neglected (velocity dispersion,...) and we get evolution equations for the cosmic density and velocity fields:

continuity equation:

$$\frac{\partial u_i}{\partial t} + \frac{\dot{a}}{a}u_i + \frac{u_j\partial_j u_i}{a} = -\frac{\partial_i\phi}{a} - \frac{\partial_j[\rho\sigma_{ij}]}{\rho a}$$

**Poisson equation:** 

**Euler equation:** 

$$\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$$

 $\frac{\partial \delta}{\partial \delta} + \frac{1}{2} \nabla \cdot \left[ (1 + \delta) \mathbf{u} \right] = 0$ 

Anisotropic dynamics within the cosmic web: Matter escapes from voids to sheets, filaments and ends up in nodes. Anisotropic dynamics within the cosmic web: Matter escapes from voids to sheets, filaments and ends up in nodes. Anisotropic dynamics within the cosmic web: Matter escapes from voids to sheets, filaments and ends up in nodes.



# Vlasov-Poisson equations: dynamics of a self-gravitating collisionless fluid

Liouville theorem:

$$\begin{bmatrix} \frac{\partial}{\partial t} + \frac{\mathbf{p}}{ma^2} \frac{\partial}{\partial \mathbf{x}} - m \nabla \phi \frac{\partial}{\partial \mathbf{p}} \end{bmatrix} f(\mathbf{x}, \mathbf{p}, \mathbf{t}) = \mathbf{0}$$
$$\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$$

**Poisson equation:** 

These highly non-linear equations can be solved using numerical simulations or analytically in some specific regimes. Exact solutions are crucial to understand the details of structure formation.

Before shell-crossing, moments>2 can be neglected (velocity dispersion,...) and we get evolution equations for the cosmic density and velocity fields:

continuity equation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \left[ (1+\delta) \mathbf{u} \right] = 0$$
$$\frac{\partial u_i}{\partial t} + \frac{\dot{a}}{a} u_i + \frac{u_j \partial_j u_i}{a} = -\frac{\partial_i \phi}{a} - \frac{\partial_j [\rho \sigma_{ij}]}{\rho a}$$

Peebles 1980; Fry 1984; Bernardeau 2002

**Poisson equation:** 

**Euler equation:** 

$$\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$$

# Vlasov-Poisson equations: dynamics of a self-gravitating collisionless fluid

Liouville theorem:

$$\begin{bmatrix} \frac{\partial}{\partial t} + \frac{\mathbf{p}}{ma^2} \frac{\partial}{\partial \mathbf{x}} - m \nabla \phi \frac{\partial}{\partial \mathbf{p}} \end{bmatrix} f(\mathbf{x}, \mathbf{p}, \mathbf{t}) = \mathbf{0}$$
$$\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$$

**Poisson equation:** 

These highly non-linear equations can be solved using numerical simulations or analytically in some specific regimes. Exact solutions are crucial to understand the details of structure formation.

Before shell-crossing, moments>2 can be neglected (velocity dispersion,...) and we get evolution equations for the cosmic density and velocity fields:

continuity equation:
$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1+\delta)\mathbf{u}] = 0$$
Euler equation: $\frac{\partial u_i}{\partial t} + \frac{\dot{a}}{a}u_i + \frac{u_j\partial_j u_i}{a} = -\frac{\partial_i \phi}{a} - \frac{\partial_j [\phi_i i_j]}{\rho a}$ Poisson equation: $\Delta \phi = 4\pi a^2 G(\rho - \bar{\rho})$ 

Peebles 1980; Fry 1984; Bernardeau 2002

### **From Eulerian to Lagrangian space**

#### **Eulerian pt of view:**

- Fixes the frame
- Fields on a grid
- ▶ δ, u
- « volume-weighted statistics »





#### Lagrangian pt of view:

- Follows the fluid
- Particules
- ▶ x=q+ψ
- « mass-weighted statistics »



### **From Eulerian to Lagrangian space**

#### **Eulerian pt of view:**

- Fixes the frame
- Fields on a grid
- ▶ δ, u
- « volume-weighted statistics »





#### Lagrangian pt of view:

- Follows the fluid
- Particules
- ▶ x=q+ψ
- « mass-weighted statistics »

![](_page_10_Picture_13.jpeg)

### Lagrangian dynamics: Zeldovich pancakes

![](_page_11_Picture_1.jpeg)

initial position

$$\mathbf{x} = \mathbf{q} + \zeta$$

final position

displacement

At linear order in the displacement, the Vlasov Poisson system reduces to

$$\nabla_{\mathbf{q}}\ddot{\zeta} + 2H\nabla_{\mathbf{q}}\dot{\zeta} = \frac{3}{2}\Omega H^2\nabla_{\mathbf{q}}\zeta$$

which has the same solution as the linear density contrast i.e

$$\zeta_{\rm ZA} = D_+(t)\zeta_+(\mathbf{q})$$

Balistic trajectories

so that the density after a Zeldovich displacement reads:

$$\rho_{ZA}(\mathbf{q},t) = \frac{\bar{\rho}}{\left|\prod_{i=1}^{3}(1-D_{+}(t)\lambda_{i})\right|} - \partial \zeta_{+}^{(i)}/\partial q_{j}$$
Anisotropic collapse of structures and formation of *caustics*!
Walls form first followed by filaments and nodes.

« l'essence de la théorie des catastrophes c'est de ramener les discontinuités apparentes à la manifestation d'une évolution lente sous-jacente. Le problème est alors de déterminer cette évolution lente qui, elle, exige en général l'introduction de nouvelles dimensions, de nouveaux paramètres. » - René Thom (1991)

#### Lagrangian dynamics: Zeldovich pancakes

![](_page_12_Picture_1.jpeg)

initial position

$$\mathbf{x} = \mathbf{q} + \zeta$$

final position

displacement

#### At linear order in the displacement, the Vlasov Poisson system reduces to

$$\nabla_{\mathbf{q}}\ddot{\zeta} + 2H\nabla_{\mathbf{q}}\dot{\zeta} = \frac{3}{2}\Omega H^2\nabla_{\mathbf{q}}\zeta$$

![](_page_12_Figure_8.jpeg)

« l'essence de la théorie des catastrophes c'est de ramener les discontinuités apparentes à la manifestation d'une évolution lente sous-jacente. Le problème est alors de déterminer cette évolution lente qui, elle, exige en général l'introduction de nouvelles dimensions, de nouveaux paramètres. » - René Thom (1991)

#### The connected cosmic web

![](_page_13_Picture_1.jpeg)

![](_page_13_Picture_2.jpeg)

Dmitri Pogosyan

Bond, Kofman, Pogosyan 1996: first *understanding* of the origin of the cosmic web.

The seeds of walls, filaments and nodes lie in the asymmetries of the primordial Gaussian random field then amplified by gravitational instability.

Rare *peaks* in the ICs will become the nodes of the cosmic web i.e rich clusters. Their initial *shear* will set the preferred directions along which correlation bridges will connect them to other nodes.

![](_page_13_Picture_7.jpeg)

Importance of peak & constrained random field theories

#### The connected cosmic web

![](_page_14_Picture_1.jpeg)

![](_page_14_Picture_2.jpeg)

Bond, Kofman, Pogosyan 1996: first *understanding* of the origin of the cosmic web.

The seeds of walls, filaments and nodes lie in the asymmetries of the primordial Gaussian random field then amplified by gravitational instability.

Rare *peaks* in the ICs will become the nodes of the cosmic web i.e rich clusters. Their initial *shear* will set the preferred directions along which correlation bridges will connect them to other nodes.

![](_page_14_Picture_6.jpeg)

Importance of peak & constrained random field theories

#### The skeleton picture

# Filaments are the field lines joining the maxima through saddle points.

cosmic web extractors (water-shedding, discrete topology, ...)
 Sousbie+08, Sousbie+11, ...\*

Iocal theory allowing for theoretical predictions for extrema counts, length of filaments, surface of voids, curvature ... which are very competitive cosmological probes! BBKS, Pogosyan+09, Gay+12, ...\*

Cosmic connectivity κ: typically, how many filaments connect to a node? sc+18

![](_page_15_Figure_5.jpeg)

#### The skeleton picture

# Filaments are the field lines joining the maxima through saddle points.

cosmic web extractors (water-shedding, discrete topology, ...)
 Sousbie+08, Sousbie+11, ...\*

Iocal theory allowing for theoretical predictions for extrema counts, length of filaments, surface of voids, curvature ... which are very competitive cosmological probes! BBKS, Pogosyan+09, Gay+12, ...\*

Cosmic connectivity κ: typically, how many filaments connect to a node? sc+18

![](_page_16_Figure_5.jpeg)

#### The skeleton picture

# Filaments are the field lines joining the maxima through saddle points.

cosmic web extractors (water-shedding, discrete topology, ...)
 Sousbie+08, Sousbie+11, ...\*

Iocal theory allowing for theoretical predictions for extrema counts, length of filaments, surface of voids, curvature ... which are very competitive cosmological probes! BBKS, Pogosyan+09, Gay+12, ...\*

Cosmic connectivity κ: typically, how many filaments connect to a node? sc+18

![](_page_17_Figure_5.jpeg)

Importance of peak & constrained random field theories

#### **Topology of the cosmic web**

![](_page_18_Picture_1.jpeg)

1940's: Kac-Rice first studied the peaks in 1D signals, with important applications in communication theory and electronic signals

![](_page_19_Picture_2.jpeg)

Applications to cosmology then follows with Doroshkevich (1970), Bardeen-Bond-Kaiser-Szalay (1986) and many others ...

![](_page_19_Figure_4.jpeg)

1957: Longuet-Higgins extended this work to the 2D case in the context of ocean surface waves (width and shape of the crests, distance between troughs, ...)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives.

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

![](_page_21_Picture_4.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{peak k}) \times (\vec{r} - \vec{r}_{peak k})_{j}$$

![](_page_22_Picture_6.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{peak k}) \times (\vec{r} - \vec{r}_{peak k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

![](_page_23_Picture_7.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{\text{peak }k}) \times (\vec{r} - \vec{r}_{\text{peak }k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

So that in the end:

$$\langle n_{\text{peak}} \rangle = \int \frac{\mathrm{d}^3 \vec{r}}{V} n_{\text{peak}}(\vec{r}) = \int \mathrm{d}x \, \mathrm{d}^3 x_i \, \mathrm{d}^6 x_{ij} P(x, x_i, x_{ij}) |\det x_{ij}| \delta_D(x_i)$$

![](_page_24_Picture_8.jpeg)

![](_page_24_Picture_9.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{\text{peak }k}) \times (\vec{r} - \vec{r}_{\text{peak }k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

So that in the end:

$$\langle n_{\text{peak}} \rangle = \int \frac{\mathrm{d}^{3}\vec{r}}{V} n_{\text{peak}}(\vec{r}) = \int \mathrm{d}x \, \mathrm{d}^{3}x_{i} \, \mathrm{d}^{6}x_{ij} P(x, x_{i}, x_{ij}) |\det x_{ij}| \delta_{D}(x_{i})$$
ergodicity!
spatial average=ensemble average

![](_page_25_Picture_8.jpeg)

![](_page_25_Picture_9.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{peak k}) \times (\vec{r} - \vec{r}_{peak k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

So that in the end:

$$\langle n_{\text{peak}} \rangle = \int \frac{\mathrm{d}^{3}\vec{r}}{V} n_{\text{peak}}(\vec{r}) = \int \mathrm{d}x \, \mathrm{d}^{3}x_{i} \, \mathrm{d}^{6}x_{ij} P(x, x_{i}, x_{ij}) |\det x_{ij}| \delta_{D}(x_{i}) \times \Theta(-\lambda_{1})$$
ergodicity!
spatial average=ensemble average

![](_page_26_Picture_9.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{peak k}) \times (\vec{r} - \vec{r}_{peak k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

So that in the end:

$$\langle n_{\text{peak}} \rangle = \int \frac{\mathrm{d}^{3}\vec{r}}{V} n_{\text{peak}}(\vec{r}) = \int \mathrm{d}x \, \mathrm{d}^{3}x_{i} \, \mathrm{d}^{6}x_{ij} P(x, x_{i}, x_{ij}) |\det x_{ij}| \delta_{D}(x_{i}) \times \Theta(-\lambda_{1})$$
ergodicity!
$$\times \Theta(x - \sigma_{0}\nu)$$
spatial average=ensemble average

![](_page_27_Picture_9.jpeg)

Let us consider a field x and its first  $x_i$  and second  $x_{ij}$  derivatives. The number density of peaks is:

$$n_{\text{peak}}(\vec{r}) = \sum_{k} \delta_D(\vec{r} - \vec{r}_{peak\,k})$$

A Taylor expansion of  $x_i$  around a peak k reads:

$$\nabla (\vec{r}) = 0 + \sum_{j} (\vec{r}_{j}) (\vec{r}_{peak k}) \times (\vec{r} - \vec{r}_{peak k})_{j}$$

which can be inverted :  $(\vec{r} - \vec{r}_{\text{peak }k}) = \mathcal{H}^{-1}(\vec{r}_{\text{peak }k}) \cdot \nabla(\vec{r})$ 

So that in the end:

$$\langle n_{\text{peak}} \rangle = \int \frac{\mathrm{d}^{3}\vec{r}}{V} n_{\text{peak}}(\vec{r}) = \int \mathrm{d}x \, \mathrm{d}^{3}x_{i} \, \mathrm{d}^{6}x_{ij} P(x, x_{i}, x_{ij}) |\det x_{ij}| \delta_{D}(x_{i}) \times \Theta(-\lambda_{1}) \\ \text{ergodicity!} \times \Theta(x - \sigma_{0}\nu) \\ \text{spatial average=ensemble average}$$

![](_page_28_Picture_9.jpeg)

#### **Peak theory: Gaussian predictions**

If the field is Gaussian (large scales/early times),  $X = (x, x_i, x_{ij})$  follows a normal distribution

$$P(x, x_i, x_{ij}) = \frac{\exp\left(-\frac{X^{\tau} \cdot C^{-1} \cdot X}{2}\right)}{\sqrt{\det(2\pi C)}}$$

where the covariance matrix C of the field, its first and second derivatives can easily be computed from the power spectrum.

E.g in 3D, once the fields are rescaled by their variance:

with spectral parameter

$$\gamma = \frac{\sigma_1^2}{\sigma_0 \sigma_2} \\ = \frac{\langle \nabla x^2 \rangle}{\sqrt{\langle x^2 \rangle \langle \Delta x^2 \rangle}}$$

#### **Peak theory: Gaussian predictions**

If the field is Gaussian (large scales/early times), the total number density of critical points then reads

![](_page_30_Figure_2.jpeg)

$$\begin{aligned} \mathbf{3D} \\ \langle n_{\max} \rangle &= \langle n_{\min} \rangle = \frac{29\sqrt{15} - 18\sqrt{10}}{1800\pi^2 R_{\star}^3} \\ \langle n_{\mathrm{sadf}} \rangle &= \langle n_{\mathrm{sadw}} \rangle = \frac{29\sqrt{15} + 18\sqrt{10}}{1800\pi^2 R_{\star}^3}, \end{aligned}$$

#### **Peak theory: Gaussian predictions**

If the field is Gaussian (large scales/early times), the total number density of critical points then reads

![](_page_31_Figure_2.jpeg)

$$\begin{aligned} \mathbf{3D} \\ \langle n_{\max} \rangle &= \langle n_{\min} \rangle = \frac{29\sqrt{15} - 18\sqrt{10}}{1800\pi^2 R_{\star}^3} \\ \langle n_{\mathrm{sadf}} \rangle &= \langle n_{\mathrm{sadw}} \rangle = \frac{29\sqrt{15} + 18\sqrt{10}}{1800\pi^2 R_{\star}^3}, \end{aligned}$$

And as a function of peak height (analytical in 2D, not in 3D) :

![](_page_31_Figure_5.jpeg)

see also Pogosyan+00, Gay+11, SC+13

**Gram-Charlier expansion** (analogous to a Taylor expansion for PDF): The moment expansion of the general PDF P(x) around a Gaussian G(x) is an Hermite expansion:

$$P(x) = G(x) \left[ 1 + \sum_{n=3}^{\infty} \frac{1}{n!} \langle x^n \rangle_{GC} H_n(x) \right]$$

to all order in non gaussianity

where Hermite polynomials are polynomials of order n in x, orthogonal wrt the Gaussian kernel G.

A similar expansion holds for  $P(x,x_i,x_{ij})$ 

see also Pogosyan+00, Gay+11, SC+13

**Gram-Charlier expansion** (analogous to a Taylor expansion for PDF): The moment expansion of the general PDF P(x) around a Gaussian G(x) is an Hermite expansion:

$$P(x) = G(x) \left[ 1 + \sum_{n=3}^{\infty} \frac{1}{n!} \langle x^n \rangle_{GC} H_n(x) \right]$$

to all order in non gaussianity

where Hermite polynomials are polynomials of order n in x, orthogonal wrt the Gaussian kernel G.

A similar expansion holds for  $P(x, x_i, x_{ij})$ and allows us to get predictions for number density of peaks to all order in non-Gaussianity once rotational invariants are used :

$$n_{\mp ---} = \frac{29\sqrt{15} \mp 18\sqrt{10}}{1800\pi^2 R_*^3} + \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3} \left( \left\langle q^2 J_1 \right\rangle - \frac{8}{21} \left\langle J_1^3 \right\rangle + \frac{10}{21} \left\langle J_1 J_2 \right\rangle \right)$$
$$n_{++\pm} = \frac{29\sqrt{15} \mp 18\sqrt{10}}{1800\pi^2 R_*^3} - \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3} \left( \left\langle q^2 J_1 \right\rangle - \frac{8}{21} \left\langle J_1^3 \right\rangle + \frac{10}{21} \left\langle J_1 J_2 \right\rangle \right)$$

see also Pogosyan+00, Gay+11, SC+13

**Gram-Charlier expansion** (analogous to a Taylor expansion for PDF): The moment expansion of the general PDF P(x) around a Gaussian G(x) is an Hermite expansion:

$$P(x) = G(x) \left[ 1 + \sum_{n=3}^{\infty} \frac{1}{n!} \langle x^n \rangle_{GC} H_n(x) \right]$$

to all order in non gaussianity

where Hermite polynomials are polynomials of order n in x, orthogonal wrt the Gaussian kernel G.

A similar expansion holds for  $P(x, x_i, x_{ij})$ and allows us to get predictions for number density of peaks to all order in non-Gaussianity once rotational invariants are used :

$$n_{\mp ---} = \frac{29\sqrt{15} \mp 18\sqrt{10}}{1800\pi^2 R_*^3} + \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3} \left( \left\langle q^2 J_1 \right\rangle - \frac{8}{21} \left\langle J_1^3 \right\rangle + \frac{10}{21} \left\langle J_1 J_2 \right\rangle \right)$$

$$n_{++\pm} = \frac{29\sqrt{15} \mp 18\sqrt{10}}{1800\pi^2 R_*^3} - \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3} \left( \left\langle q^2 J_1 \right\rangle - \frac{8}{21} \left\langle J_1^3 \right\rangle + \frac{10}{21} \left\langle J_1 J_2 \right\rangle \right)$$

Those cumulants can be predicted from PT  $\,\propto\,\sigma$ 

#### **Peak theory: Non-Gaussian predictions**

![](_page_35_Figure_1.jpeg)

Those cumulants can be predicted from PT  $\,\propto\,\sigma$ 

### Peak theory: clustering (i.e 2pt stat)

Same ideas can be used to also predict the clustering of peaks by means of their 2 point correlation function (higher order statistics are also possible although not much investigated so far):

![](_page_36_Figure_2.jpeg)

Alternative to the usual use of N-point correlation functions / poly-spectra,... which is :

- independent from bias (M/L ratio)
- easier to measure in the data (less sensitive to masks,...), more robust

Because topology is about shapes, connectivity, holes,... and is *invariant* under *continuous* deformation (stretching, twisting, bending...).

![](_page_37_Picture_5.jpeg)

# **Topology of excursion sets**

![](_page_38_Figure_1.jpeg)

# **Topology of excursion sets**

![](_page_39_Figure_1.jpeg)

> Minkowski functionals (topological invariants):

d+1 MFs in d dimensions.

Mathematical genus in 2D = number of handles/holes (max number of cuttings along closed curves without disconnecting the surface)

![](_page_40_Figure_4.jpeg)

![](_page_40_Picture_5.jpeg)

![](_page_40_Figure_6.jpeg)

upcrossing maxima=-1

> extrema counts

upcrossing minima=+1

> Minkowski functionals (topological invariants):

d+1 MFs in d dimensions.

Mathematical genus in 2D = number of handles/holes (max number of cuttings along closed curves without disconnecting the surface)

![](_page_41_Figure_4.jpeg)

This is a topological invariant: two surfaces are homeomorphic if they have the same genus.

![](_page_41_Picture_6.jpeg)

![](_page_41_Figure_7.jpeg)

extrema

counts

upcrossing minima=+1

> Minkowski functionals (topological invariants):

d+1 MFs in d dimensions.

Mathematical genus in 2D = number of handles/holes (max number of cuttings along closed curves without disconnecting the surface)

![](_page_42_Figure_4.jpeg)

This is a topological invariant: two surfaces are homeomorphic if they have the same genus.

In ND, we define the **Euler-Poincaré characteristic** (in 2D, =2-2g) as the alternating sum of Betti numbers:

$$\chi = \sum_{i} (-1)^i b_i$$

where  $b_i$  is its rank of the i-th homology group ( $b_0$ =number of connected components,  $b_1$ =circular holes,  $b_2$ =cavities,...). **Gauss-Bonnet theorem:** *χ* is the integral of the Gaussian curvature Morse theory: it is the alternating sum of extrema.

The Euler characteristic obeys: additivity, motion invariance and conditional continuity, it is one of the MF.

![](_page_42_Picture_10.jpeg)

![](_page_42_Figure_11.jpeg)

upcrossing minima=+1

extrema

counts

Minkowski functionals (topological invariants): d+1 MFs in d dimensions: Euler-Poincaré characteristic and?? in 2D: length of isocontour + encompassed volume in 3D: surface of isocontour+encompassed volume+integrated mean curvature

geometrical estimators and critical sets: peak/saddle/void counts length of filaments surface of walls

• • •

![](_page_43_Picture_4.jpeg)

![](_page_43_Picture_5.jpeg)

## **Euler-Poincaré characteristic**

$$\chi_{3\mathrm{D}}(\nu) = -\int P(x, x_i, x_{ij}) \delta_{\mathrm{D}}(x_i) \det x_{ij} \Theta(x - \sigma_0 \nu)$$

Using a Gram-Charlier expansion and invariant variables, on can get a prediction to all orders in non-Gaussianity

$$\begin{split} \chi(\nu) &= \frac{1}{2} \mathrm{Erfc} \left(\frac{\nu}{\sqrt{2}}\right) \chi(-\infty) + \frac{1}{27R_*^3} \left(\frac{3}{2\pi}\right)^{3/2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\nu^2}{2}\right) \left[\gamma^3 H_2\left(\nu\right) + \right. \\ &+ \sum_{n=3}^{\infty} \sum_{i,j,k}^{i+2j+k=n} \frac{(-1)^k}{i! \, j!} \left(-\frac{3}{2}\right)^j \left\langle \zeta^i q^{2j} J_1{}^k \right\rangle_{\mathrm{GC}} \left(1-\gamma^2\right)^{i/2} \sum_{s=0}^{\min(3,k)} \frac{3! \gamma^{k+3-2s}}{s! (3-s)! (k-s)!} H_{i+k+2-2s}\left(\nu\right) \\ &+ \sum_{n=3}^{\infty} \sum_{i,j,k}^{i+2j+k+2=n} \frac{(-1)^{k+1} 3}{i! \, j!} \left(-\frac{3}{2}\right)^j \left\langle \zeta^i q^{2j} J_1{}^k J_2 \right\rangle_{\mathrm{GC}} \left(1-\gamma^2\right)^{i/2} \sum_{s=0}^{\min(1,k)} \frac{\gamma^{k+1-2s}}{(k-s)!} H_{i+k-2s}\left(\nu\right) \\ &+ \sum_{n=3}^{\infty} \sum_{i,j,k}^{i+2j+k+3=n} \frac{(-1)^{k+1} 3}{i! \, j!} \left(-\frac{3}{2}\right)^j \left\langle \zeta^i q^{2j} J_1{}^k J_2 \right\rangle_{\mathrm{GC}} \left(1-\gamma^2\right)^{i/2} \gamma^k H_{i+k-1}\left(\nu\right) \right] \end{split}$$

### **Euler-Poincaré characteristic**

$$\chi_{3D}(\nu) = -\int P(x, x_i, x_{ij})\delta_D(x_i) \det x_{ij}\Theta(x - \sigma_0\nu)$$

Using a Gram-Charlier expansion and invariant variables, on can get a prediction to all orders in non-Gaussianity

![](_page_45_Figure_3.jpeg)

#### **Topology of the cosmic web**

![](_page_46_Picture_1.jpeg)

Birth and growth of the cosmic web
 Random fields, Peak theory, topology
 Cosmic connectivity

A 10-year long work with Dmitri Pogosyan (UAlberta) & Christophe Pichon (IAP) Codis, Pogosyan, Pichon, 2018, MNRAS, 479, 973

# **Global connectivity for GRF**

![](_page_47_Figure_1.jpeg)

Can we predict the mean connectivity?

# **Global connectivity for GRF: theory**

Because each filament goes through one and only one saddle pt, on average:

$$\begin{aligned} \langle \kappa \rangle &= \frac{2\bar{n}_{\rm sad}}{\bar{n}_{\rm max}} \\ &= 4 & \text{in 2D GRF} \\ &= \frac{2\left(1057 + 348\sqrt{6}\right)}{625} \approx 6.11 \text{ in 3D GRF} \end{aligned}$$

### **Global connectivity for GRF: theory**

Because each filament goes through one and only one saddle pt, on average:

$$\begin{aligned} \langle \kappa \rangle &= \frac{2\bar{n}_{\rm sad}}{\bar{n}_{\rm max}} \\ &= 4 & \text{in 2D GRF} \\ &= \frac{2\left(1057 + 348\sqrt{6}\right)}{625} \approx 6.11 \text{ in 3D GRF} \end{aligned}$$

In d dimensions, we relied on numerical integrations:

![](_page_49_Figure_4.jpeg)

# **GRF connectivity: dependence with peak height**

Full distribution of connectivity:

![](_page_50_Figure_2.jpeg)

Dependence with peak height:

The higher the peak, the more connected

# **Global connectivity: evolution with cosmic time**

![](_page_51_Figure_1.jpeg)

![](_page_51_Figure_2.jpeg)

Filaments merge in a cosmology-dependent way!

# **Global connectivity: evolution with cosmic time**

![](_page_52_Figure_1.jpeg)

Measurement in cosmological simulations:

Filaments merge in a cosmology-dependent way!

Predictions:

Using a Gram Charlier expansion, one can get prediction at arbitrary order in NG

$$\langle \kappa \rangle = \kappa^{\mathbf{G}} \left( 1 + \sum_{i \ge 1} \kappa^{(i)} \sigma_0^i \right)$$

With

$$\kappa^{\rm G} = 2 \times \frac{29\sqrt{3} + 18\sqrt{2}}{29\sqrt{3} - 18\sqrt{2}} \approx 6.11$$

$$\kappa^{(1)} = \frac{4\sqrt{3}}{35\sqrt{\pi}\sigma_0} \left(8\left\langle J_1^3 \right\rangle - 10\left\langle J_1 J_2 \right\rangle - 21\left\langle q^2 J_1 \right\rangle\right)$$

# Local multiplicity and bifurcation points

For galaxy formation, what matters most is how many filament connect **locally** onto a galaxy. At small enough scale, a peak is always **ellipsoidal** so that only two branches of filament stick out. Then those branches **bifurcate**. Some bifurcations appear so close to the peak that they are physically irrelevant. Hence we will define the **multiplicity** as the local number of filaments.

![](_page_53_Figure_2.jpeg)

![](_page_53_Figure_3.jpeg)

# Local multiplicity and bifurcation points

For galaxy formation, what matters most is how many filament connect **locally** onto a galaxy. At small enough scale, a peak is always **ellipsoidal** so that only two branches of filament stick out. Then those branches **bifurcate**. Some bifurcations appear so close to the peak that they are physically irrelevant. Hence we will define the **multiplicity** as the local number of filaments.

![](_page_54_Figure_2.jpeg)

![](_page_54_Figure_3.jpeg)

### Local multiplicity and bifurcation points

For galaxy formation, what matters most is how many filament connect **locally** onto a galaxy. At small enough scale, a peak is always **ellipsoidal** so that only two branches of filament stick out. Then those branches **bifurcate**. Some bifurcations appear so close to the peak that they are physically irrelevant. Hence we will define the **multiplicity** as the local number of filaments.

![](_page_55_Figure_2.jpeg)

# Local multiplicity

The denser the environment, the higher the multiplicity therefore bringing less coherent angular momentum and generating more ellipsoidal galaxies?

![](_page_56_Figure_2.jpeg)

# Local multiplicity

The denser the environment, the higher the multiplicity therefore bringing less coherent angular momentum and generating more ellipsoidal galaxies?

![](_page_57_Figure_2.jpeg)

Let us count filament crossings at a sphere of radius R around the central peak...

![](_page_58_Figure_2.jpeg)

Let us count filament crossings at a sphere of radius R around the central peak...

![](_page_59_Figure_2.jpeg)

Let us count filament crossings at a sphere of radius R around the central peak...

![](_page_60_Figure_2.jpeg)

![](_page_61_Figure_1.jpeg)

![](_page_62_Figure_1.jpeg)

![](_page_63_Figure_1.jpeg)

But how dense are those filaments?

![](_page_64_Figure_1.jpeg)

Typically, two to three dense filaments dominate and therefore define a plane of accretion... in agreement with numerical simulation (Danovich+12) and observations of plane os satellites around galaxies.

#### Conclusion

- Peak and constrained random field theories are paramount to understand the birth and growth of the cosmic web
- Many analytical results can be obtained in the weakly non-linear regime

horizon-AGN

- The topology and geometry of the cosmic web carries important cosmological information and is key for galaxy evolution.
- In particular, we now have a precise understanding of the connectivity of the cosmic web (the cosmic crystal)

![](_page_65_Picture_5.jpeg)

![](_page_66_Picture_1.jpeg)